Mathematics for Materials

Problem Set A for 'Vectors, Matrices...' Course
(overall 3rd set in the Michaelmas Term maths provision)

These first 4 questions should be do-able after the first lecture of the course.

1. For each of the following pairs of points A and B find:
   (i) The vector from A to B, $\overrightarrow{AB}$ in $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ form
   (ii) The vector $\overrightarrow{AB}$ in $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ form
   (iii) The magnitude, $|\overrightarrow{\boldsymbol{AB}}|$
   (iv) The unit vector of $\overrightarrow{\boldsymbol{AB}}$
   (v) The direction cosines of $\overrightarrow{\boldsymbol{AB}}$

   (a) $A = (0, 0, 0)$ and $B = (2, 3, 0)$. (b) $A = (1, -3, 2)$ and $B = (0, 1, 0)$.
   (c) $A = (-1, -1, 3)$ and $B = (2, 4, -2)$.

2. Consider the regular hexagon $OABCDE$ with O at the origin, where A has position vector $\mathbf{a}$, and B has position vector $\mathbf{a} + \mathbf{b}$.
   (a) Express the position of the other three vertices in terms of $\mathbf{a}$ and $\mathbf{b}$. *Hint: Draw a sketch and solve the problem graphically.*
   (b) If someone moves around the hexagon (from O to A to B etc.), what are the expressions for each of the six moves they make in terms of $\mathbf{a}$ and $\mathbf{b}$?

3. You are out walking when one of the other people in the group is injured and needs medical assistance. Consulting a map, you decide that you are at 5 km east, 3 km south (relative to the centre of the map), and at an elevation of 200 m. The two closest towns are Rushton at 1 km east, 3 km north, elevation 800 m and Charmoor at 2 km west, 1 km south, elevation 500 m. Clearly showing your reasoning, answer the following:
   (a) What compass bearing should you follow to reach each town?
   (b) Assuming you walk in a straight line, which town is closer?
   (c) If one kilometre of net vertical change costs you five times as much energy as one kilometre of lateral movement, which town can you reach more efficiently?

4. In triangle ABC, D is the mid-point of the side AB, E is the mid-point of side BC, and X is the intersection of CD and AE. Using vectors, find the ratio of the length of CX to the length of XD and similarly find the ratio of AX to XE. *Hints: Define two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *to coincide with two sides of the triangle, and then express everything else in terms of those vectors. The ratio you are looking for is a pure number, not depending on the lengths of the sides (so it's the same for all triangles) and point* X *divides* CD *in the* same ratio *as it divides* AE*. Lookup "centroid" for more information.*

The remaining questions should be do-able after the first week of the course is complete

5. (a) Use a diagram to explain why $\mathbf{r} = \lambda\mathbf{a} + \mathbf{b}$ describes a line.
   (b) Find the vector equation of the line containing the points $(2, -2, 5)$ and $(0, -3, 1)$.
   (c) Find the vector equation of the line containing the points $2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$ and $-3\mathbf{u} + \mathbf{w}$ where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are non-coplanar vectors. *N.B. You should not assume that* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are orthogonal or that they are unit vectors.*
6. (a) Use a diagram to explain why $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c}$ describes a plane.
   (b) Find the vector equation of the plane containing the points $(3, -1, 3)$, $(1, 1, -2)$, and $(0, 2, 4)$.
   (c) A plane contains two points whose position vectors from the origin are $\overrightarrow{OA} = \mathbf{u} + \mathbf{v} + \mathbf{w}$ and $\overrightarrow{OB} = 2\mathbf{u} - \mathbf{v} - \mathbf{w}$. Furthermore, the vector $-\mathbf{u} + 2\mathbf{v}$ is parallel to the plane. Find the vector equation for this plane, given that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are non-coplanar vectors.
7. (a) Using a sketch, explain the geometrical meaning of the equation of the dot product, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.
   (b) Explain how you can use the dot product to find the part of $\mathbf{a}$ parallel to $\mathbf{b}$.
   (c) Resolve the vector $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ into two vectors, one parallel to $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ and one perpendicular.
8. (a) Use a diagram to explain why $\mathbf{r} \cdot \mathbf{n} = d$ describes a plane. What meaning does d have if $\mathbf{n}$ is a unit vector?
   (b) A plane contains the point $(3, -1, 3)$, and vector $\mathbf{n} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ is normal to the plane. What is the shortest distance to point $(4, 3, 1)$?
9. (a) Write down and explain the vector equation of a circle using a diagram. Explain how this relates to the Cartesian equation for a circle.
   (b) Write down the vector and Cartesian equations of a sphere, centre $(1, 2, -4)$ and radius 5.
10. (a) Vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. What are $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{a}$, $|\mathbf{a} \times \mathbf{b}|$, and $|\mathbf{b} \times \mathbf{b}|$?
    (b) Let the components of $\mathbf{a}$ and $\mathbf{b}$ be a_x, a_y, a_z and b_x, b_y, b_z respectively. Show that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and to $\mathbf{b}$.
11. (a) Using a sketch, explain the geometrical meaning of the equation of the cross product, $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.
    (b) Using vectors, find the area of a parallelogram having one corner at the origin $(0, 0, 0)$ and the adjacent two corners at $(1, 0, 3)$ and $(4, -2, -2)$.
    (c) Using vectors, find the area of the triangle with corners at $(0, 0, 0)$, $(-2, 3, 2)$ and $(1, 1, -4)$.
12. (a) Transform the plane equation $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ into the form $\mathbf{r} \cdot \mathbf{n} = d$, and hence find the distance to the point $(3, 5, 7)$. Explain your working.
    (b) Do the lines $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ define a plane? Justify your answer.
    (c) $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are non-coplanar vectors. Does the line $\mathbf{r} = 5\mathbf{a} - 8\mathbf{b} + 6\mathbf{c} + \gamma(-2\mathbf{a} - 16\mathbf{b} + 13\mathbf{c})$ lie in the plane $\mathbf{r} = 3\mathbf{a} + \mathbf{c} + \lambda(\mathbf{a} - 2\mathbf{b} + \mathbf{c}) + \mu(-\mathbf{a} - 2\mathbf{b} + 2\mathbf{c})$? Justify your answer.
13. What is the shortest distance between point $(2, 0, -3)$ and the line $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$? Use a sketch to explain your working.