

Mathematics for Materials

Problem Set C for the ‘Vectors, Matrices...’ Course

(overall 5th set in the Michaelmas Term maths provision)

Problems for Vector Calculus

1. Find the vector field $\mathbf{F} = \text{grad } S$, also written $\mathbf{F} = \nabla S$, for each of the following scalar fields S . Remember that if there is no z component, then it is just a 2D problem and the $\mathbf{k}$ component of the vector field doesn't exist. Make a sketch for each case to show the resulting vector field; for 3D cases, simply show a slice e.g. $z = 0$.

(a) $S = 2x^2 - y^2$

(b) $S = 1/(x^2 + y^2 + z^2)$

(c) $S = 1/(x^2 + y^2 + z^2)^{\frac{1}{2}}$

(d) $S = x + y + z$

- (e) Guess a sketch of the following (don't work through the maths):

$$S = \frac{1}{((x-1)^2 + y^2 + z^2)^{\frac{1}{2}}} - \frac{1}{((x+1)^2 + y^2 + z^2)^{\frac{1}{2}}}$$

If the sign between the two terms changed from $-$ to $+$ how would that change the result?

2. For each of the following vector fields $\mathbf{F}$, find the divergence $Q(x, y, z) = \nabla \cdot \mathbf{F}$,

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) $\mathbf{F} = x^2y\mathbf{i} + y^3\mathbf{j} - z^{\frac{1}{2}}\mathbf{k}$

(c) $\mathbf{F} = z^{\frac{1}{2}}\mathbf{i} + x^3\mathbf{j} + y^2\mathbf{k}$

(d) $\mathbf{F} = \sin(x)\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$

- 3.* For the gravitational force field due to a point mass M , and felt by a test mass m i.e.

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm\mathbf{r}}{|\mathbf{r}|^3}$$

show that the divergence $\nabla \cdot \mathbf{F}(\mathbf{r})$ is zero for any $|\mathbf{r}| > 0$.

4. For the following two path integrals, in each case (a) check for path-independence by examining whether $\mathbf{F}$ can be written as $\mathbf{F} = \nabla S$ for some scalar function S , and then (b) evaluate the integral along a suitable path

(a) $\int_C \mathbf{F} d\mathbf{l}$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and one path is from point $(x, y, z) = (1, 0, 0)$ to point $(-1, 0, 0)$ along the half-circle that passes through point $(0, 1, 0)$.

(b) $\int_C \mathbf{F} d\mathbf{l}$ where $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} + z^2\mathbf{k}$ and one path is from point $(x, y, z) = (0, 0, 0)$ to $(0, 0, 5)$ in a straight line.

- 5.**A certain charged object produces a spherically-symmetric electric force field; the force-per-Coulomb on a test charge located at position vector $\mathbf{r}$ would be:

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q\mathbf{r}}{|\mathbf{r}|^3}$$

By following the description in the notes, use Gauss' Law and perform a surface integral over a closed sphere of radius r in order to confirm that the total charge within the sphere is Q .

Matrix problems

- Demonstrate that matrix multiplication is not commutative (that in general $\mathbf{AB} \neq \mathbf{BA}$).
- For each of the following, draw a sketch showing how the deformation affects a unit cube. Explain clearly how the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vectors are affected by the transformation and mark them on your sketch. Use your sketch to write down the deformation matrix. *Hint: Make sure your coordinate system is right-handed.*
 - Normal strain of magnitude ε along the initial $\hat{\mathbf{j}}$ axis
 - Simple** shear of magnitude σ , with the deformation in the x - y plane and in the x direction
 - Pure** shear of magnitude τ in the y - z plane
- Consider the operation of rotating any general vector anticlockwise by ϕ about the y -axis. Draw a sketch showing a vector before and after rotation, and use trigonometry to show how the components of the old vector relate to the new vector. Thus use your sketch to write down the rotation matrix. *Hint: The rotation direction is defined as you look down the positive axis of rotation towards the origin.*
- Sketch a cube before and after the following deformation: tensile strain of 100% along the z axis and compressive strain of 50% along the y axis, followed by rotation of 45° anticlockwise about the z axis.
 - Sketch a cube before and after: rotation of 45° anticlockwise about the z axis, followed by tensile strain of 100% along the z axis and compressive strain of 50% along the y axis.
 - Find the complete transformation matrix for each of the above.
 - Carefully verify your matrices are correct by checking the transformation of key position vectors.
- What transformations do the given matrices represent?

$$(a) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$(b) \begin{pmatrix} 1 & (1+\varepsilon)\sigma & 0 \\ 0 & 1+\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- The crystal defined by the following basis vectors: $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ belongs to one of the six crystal systems. Classify the crystal and justify your answer.
 - The crystal is exposed to heat and pressure causing it to undergo a phase transition. The deformation can be described by

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & 2 \\ 1 & 0 & \frac{2}{3} \end{pmatrix}$$

Classify the new crystal.

- Transform a general vector in the $z = 0$ plane and describe the shape and position of this plane after the deformation.