Mathematics for Materials

Problem Set D for the 'Vectors, Matrices...' Course
(overall 6th set in the Michaelmas Term maths provision)

Online Problems

The following questions 1 to 5 can be answered using the **online system**. They are included on this sheet for reference only.

1. What is the value of the determinant $\begin{vmatrix} 1 & 5 \\ -3 & 2 \end{vmatrix}$? (a) 11 (b) 17 (c) 3 (d) -13.
2. What is the value of the determinant $\begin{vmatrix} 2 & -7 \\ -7 & 2 \end{vmatrix}$? (a) 45 (b) -5 (c) 53 (d) -45.
3. What is the value of the determinant $\begin{vmatrix} 0 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 5 & 1 \end{vmatrix}$? (a) -22 (b) 22 (c) 10 (d) 0.
4. What is the value of the determinant $\begin{vmatrix} 6 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 4 & -3 \end{vmatrix}$? (a) 7 (b) 43 (c) -5 (d) -13.
5. What is the value of the determinant $\begin{vmatrix} 1 & 0 & 0 & 2 \\ -1 & 3 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 5 & -3 & 2 \end{vmatrix}$? (a) -36 (b) 18 (c) 7 (d) 14.

Hint: Remember you can add/subtract rows from one another, then evaluate the determinant.

Pen and Paper Problems

1. (a) Show that $\begin{vmatrix} a & b & c \\ d & f & g \\ h & j & k \end{vmatrix}$ represents the volume of a three dimensional shape. *N.B. Figure out the volume by some means* other *than a determinant, and compare!*

   (b) Show that $|\mathbf{A}^{\mathbf{T}}| = |\mathbf{A}|$, i.e. that $\begin{pmatrix} a & d & h \\ b & f & j \\ c & g & k \end{pmatrix}$ spans the same volume as $\begin{pmatrix} a & b & c \\ d & f & g \\ h & j & k \end{pmatrix}$.

2. Using your results from question 1, show geometrically how the determinant of a 3×3 matrix is affected by the following:
   (a) Any two rows are swapped
   (b) A multiple of one row is added to another row
   (c) A row is multiplied by a constant
   (d) A row contains only zeros
   (e) One row is proportional to another row
   (f) What happens to the above properties if we consider columns instead of rows?

3. Using the properties of determinants, simplify and evaluate these determinants. Do not use a calculator.

   (a) $\begin{vmatrix} 30 & -150 & -45 \\ 51 & 85 & 306 \\ 77 & 0 & 462 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & x & -1 \\ 8x & 4x^2 & -12xy \\ y^2 & x & 3 \end{vmatrix}$

4. (a) Evaluate

$$\begin{vmatrix} 4 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 7 & -5 & -3 & 0 & 0 \\ 16 & -4 & 8 & 2 & 0 \\ 1 & 1 & 2 & 3 & 5 \end{vmatrix}$$

   (b) The above matrix is called a *lower triangular matrix*. Its determinant can be found by what simple calculation?

   (c) Use your answer to (b) to evaluate the following (an *upper triangular matrix*):

$$\begin{vmatrix} 2 & 37 & 51 & 412 & -73 \\ 0 & 3 & 82 & -115 & 345 \\ 0 & 0 & 5 & 666 & 382 \\ 0 & 0 & 0 & 7 & 525 \\ 0 & 0 & 0 & 0 & 11 \end{vmatrix}$$

   (d) Use the properties of determinants to convert the following to an upper triangular matrix and thus evaluate its determinant:

$$\begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 \\ 0 & 4 & 5 & 6 & 0 \\ 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 8 & 9 \end{vmatrix}$$

5. Find the inverse of each of the followintg, using either the method of cofactors or Gaussian elimination:

   (a) $\begin{pmatrix} 3 & -7 \\ 5 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 & -2 \\ 8 & 4 & 1 \\ 6 & -3 & -5 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 1 & -1 \\ 0 & 6 & 2 \\ -1 & 4 & 9 \end{pmatrix}$

6. If the elements of a transformation matrix are constants, it transforms a cube into a parallelepiped. For a cube of volume V initially aligned along the x, y, z axis, find the volume of the parallelepiped formed after the following transformations:

   (a) $\begin{pmatrix} 1+\varepsilon_x & 0 & 0 \\ 0 & 1+\varepsilon_y & 0 \\ 0 & 0 & 1+\varepsilon_z \end{pmatrix}$

   (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   (c) A pure shear of magnitude τ in the $x-y$ plane in the x and y directions

   (d) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

   (e) Which types of transformation change the cube's volume, and which do not?

7. Find the matrix for the transformation that 'undoes' each of the following and explain your answers:

   (a) A rotation of θ clockwise about the y axis

   (b) Pure shear of magnitude τ on the x-z plane accompanied by 50% compression in the y direction

8. Solve the following system of simultaneous equations using each of the following methods:

$$\begin{aligned} 2x - y + 2z &= 6 \\ 4x + 3y - 7z &= -11 \\ x + y - 3z &= -6 \end{aligned}$$

(a) Cramer's rule

(b) Gaussian elimination

(c) Finding the inverse matrix using cofactors