

## Mathematics for Materials

### Problem Set E for the 'Vectors, Matrices...' Course

(overall 7th set in the Michaelmas Term maths provision)

#### Online Quiz Questions

1. The vector  $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$  is an eigenvector of the matrix  $\mathbf{M} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ . What is the eigenvalue that goes with it? (a) -3. (b) -1. (c) 0. (d) 3.
2. One of the other two eigenvalues of the same matrix  $\mathbf{M} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$  is 2. What is the eigenvector that goes with that value?  
(a)  $\frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ -1 \\ \sqrt{5} \end{pmatrix}$ . (b)  $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ . (c)  $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ . (d)  $\begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$ .
3. What are the eigenvalues of  $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$ ?  
(a) 6, 1. (b) 8, -1. (c) 4, 3. (d)  $\frac{1}{2}(7 + \sqrt{17})$ ,  $\frac{1}{2}(7 - \sqrt{17})$ .
4. For the same matrix  $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$  what are the eigenvectors?  
(a)  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$ . (b)  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ . (c)  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . (d)  $\begin{pmatrix} 3 + \sqrt{17} \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 3 - \sqrt{17} \\ 4 \end{pmatrix}$ .
5. What are the eigenvalues of  $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ ?  
(a) -3, 4, 1. (b) 3, 1, 2. (c) 0, 1, 2. (d) 4, 2, 1.
6. For the same matrix  $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ , what are the three eigenvectors? (a)  $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ .  
(b)  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -12 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ . (c)  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ . (d)  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ .

#### Pen and Paper Problems

1. Consider a coordinate system rotated clockwise by  $\phi$  about the  $x_2$  axis.
  - (a) Draw a sketch showing how a vector's components in the old coordinate system relate to its components in the new system
  - (b) Use your sketch to write down the rotation matrix.
  - (c) How is this rotation matrix related to the rotation matrix for the rotation of a vector clockwise by  $\phi$  about the  $x_2$  axis?
2. Consider Cartesian axes  $x_1, x_2, x_3$  rotated anticlockwise by  $60^\circ$  about the  $x_1$  axis to produce new axes  $x'_1, x'_2, x'_3$ .
  - (a) Write down the rotation matrix relating the old and new axes.

(b) The vector  $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$  is referenced to the old axes. What are the components of this vector referenced to the new axes?

(c) The transformation matrix  $\boldsymbol{\sigma} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is referenced to the old axes. What are the components of this matrix referenced to the new axes?

3. Find the eigenvalues and normalised eigenvectors of the following matrices and verify that the trace of the matrix is equal to the sum of the eigenvalues:

(a)  $\begin{pmatrix} 13 & 5 \\ 2 & 4 \end{pmatrix}$       (b)  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$       (c)  $\begin{pmatrix} 5 & 2 & 4 \\ -3 & 6 & 2 \\ 3 & -3 & 1 \end{pmatrix}$       (d)  $\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$

*Hint for (c): Factorise the first part of your expression for the determinant and see if any of those three factors can be extracted from the other two parts.*

4. A symmetric matrix  $\mathbf{S}$  is a matrix that has the property  $\mathbf{S} = \mathbf{S}^T$ . Symmetric matrices also have the property that their eigenvectors are orthogonal. Verify that your eigenvectors for the symmetric matrix in Question 3 are orthogonal.

5. It is often useful to write a matrix representing a transformation or some physical property of a crystal in diagonal form, where the only non-zero elements lie along the main diagonal. This can be done by finding the eigenvectors of given matrix and using them to construct a rotation matrix  $\mathbf{R}$  that rotates the coordinate system such that  $\mathbf{D} = \mathbf{RMR}^T$  where  $\mathbf{M}$  is the original matrix and  $\mathbf{D}$  is the diagonal form.

(a) Find the eigenvalues and eigenvectors of  $\mathbf{M} = \begin{pmatrix} 10 & -2 & -2 \\ -2 & 13 & -5 \\ -2 & -5 & 13 \end{pmatrix}$

(b) Confirm that the trace and eigenvectors have the properties you would expect.

(c) Compute  $\mathbf{D} = \mathbf{RMR}^T$ , where  $\mathbf{R} = \begin{pmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ e_{3x} & e_{3y} & e_{3z} \end{pmatrix}$

where  $e_{ij}$  refers to the  $j$  component of the  $i^{th}$  normalised eigenvector. How does the order of the eigenvectors in  $\mathbf{R}$  affect  $\mathbf{D}$ ?

(d) Using the definitions of  $\mathbf{R}$  and  $\mathbf{D}$  and your results from (c), explain why the eigenvalues and eigenvectors of a matrix are invariant and why the eigenvalues sum to the trace of the original matrix. *Hint: What does it mean for a vector to be invariant?*

(e) By inspection, what are the eigenvalues and eigenvectors of  $\mathbf{N} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & 4 \end{pmatrix}$

6. (a) Find  $\mathbf{R}$  and  $\mathbf{D}$  such that  $\mathbf{D} = \mathbf{RMR}^T$  where  $\mathbf{D}$  is a diagonal matrix and  $\mathbf{M} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ .

(b)  $\mathbf{R}$  in part (a) represents what rotation? *Hint: If you don't recognise the rotation in  $\mathbf{R}$  consider your answer to 5c.*

(c) Sketch the effect of deformation  $\mathbf{M}$  on a cube initially aligned with the  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ , and  $\hat{\mathbf{k}}$  vectors.

(d) Sketch the effect of deformation  $\mathbf{M}$  on a cube initially aligned with the eigenvectors.

(e) The cubes in (c) and (d) have undergone the same deformation - why are they now different shapes? Use your answers to (c) and (d) to explain the physical meaning of eigenvalues

and eigenvectors in the case of a deformation matrix.

7. (Optional) There are certain features of a matrix that do not depend on our choice of coordinate system — these features are said to be *invariant*.

(a) Argue that

$$|x\mathbf{I} - \mathbf{M}| = |x\mathbf{I} - \mathbf{D}^{-1}\mathbf{M}\mathbf{D}|$$

where  $\mathbf{I}$  is the identity,  $x$  is just a scalar, and both  $\mathbf{M}$  and  $\mathbf{D}$  are matrices. In other words, you are showing that the expression  $|x\mathbf{I} - \mathbf{M}|$  does not change when  $\mathbf{M}$  is replaced with a *similar* matrix – i.e. the same matrix after a change of basis.

- (b) Argue that, in general, when you write out the equation represented by  $|\lambda\mathbf{I} - \mathbf{M}| = 0$ , you will obtain a polynomial of the form  $(-1)^n\lambda^n + (-1)^{n-1}C_1\lambda^{n-1} + \dots - C_{n-1}\lambda + C_n = 0$ , if the original matrix  $\mathbf{M}$  is size  $n \times n$ . Here the symbols  $C_i$  simply represent some set of coefficients. This is called the characteristic polynomial.
- (c) Bearing in mind your answers to the first two parts of this question, explain why the coefficients  $C_i$  of the characteristic equation do not change when  $\mathbf{M} \rightarrow -\mathbf{D}^{-1}\mathbf{M}\mathbf{D}$ .
- (d) Show that  $C_1$  is actually the *trace* of the matrix, and is also equal to the sum of the eigenvalues. The coefficient  $C_n$  is something you should recognise, too, what is it?