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1. Basic concepts for Vectors

This first set of notes reviews the fundamental concepts that we'll need in order to work with vectors. We will begin by asking “what is a vector”, and then we'll look at how vectors can be written down, how they can be manipulated, and what we can use them to do. Readers who've done A-Level maths will find most, if not all, of this familiar.

1.1 Scalar quantities

Most physical quantities that you will have met previously are **scalars**. This means that they can be described by a single number. Examples include **mass**, **temperature** and **charge**. The last of these, charge, is an example of a quantity that requires negative numbers as well as positive; but crucially there is just a single number required. We can of course add two scalar quantities, subtract one from another, or even multiply (as when computing an area from linear measurements) and divide to find ratios.

1.2 Vector quantities

Not all physical quantities can be described satisfactorily with a single number. Some quantities have both a **magnitude** and a **direction**. List some examples here:

Force,

How can we describe such quantities? One natural method would be to think of a directed line, or arrow, pointing in the appropriate direction and having a length which is proportional to the magnitude of the quantity. This is precisely how we will think of a vector.

Because a vector is a more complex object than a simple scalar, we need a notation that makes it immediately obvious when we are talking about a vector. There are two conventions: either to write a vector with a bold symbol, like **a**, or to underline it, a. In fact it looks best to use bold when presenting equations in a typeset form (as in these notes), but for written work it looks better to use the underlining convention. So, you might attempt a question which is described on the sheet by using bold notation, but when you actually work it through you should underline instead.

1.3 Cartesian system

In order for us to be able to talk in definite terms about the direction of a vector, we must have some kind of frame of reference – something with respect to which the vectors are orientated. We will use the standard Cartesian picture of x , y and z axes.

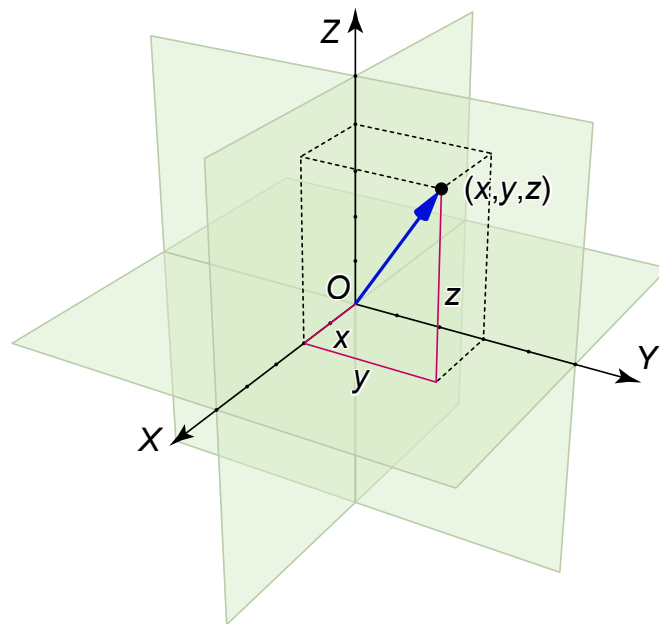


Figure 1.1: The Cartesian coordinate system, for specifying a vector.

As soon as we agree to use this coordinate system, it is obvious that we are going to need three numbers to specify a vector (well, in a three dimensional space of course). We will write the vector using the spatial coordinates of the point at its end – or more properly, the displacement along x , y and z needed to traverse the vector. Usually we will write in *column notation*. You may also see vectors in *row notation*; this looks like the notation for a *point* in Cartesian space, although sometimes square brackets are used $[]$. Be aware that there is a specific notation used for vectors in Crystallography, which we'll come to in due course. The vector $\mathbf{v}$ in the figure can be written thus:

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \leftarrow \text{column notation.} \qquad \begin{matrix} (x, y, z) \text{ or } [x, y, z] \\ \nearrow \\ \text{row notation.} \end{matrix}$$

1.4 Magnitude of a vector

Now that we have a coordinate system we are able to compute the magnitude of a vector – that is, how long it is. We recall from Pythagoras that if we have a point in the two-dimensional $x - y$ plane specified by $x = A$ and $y = B$, then the distance of that point from the origin is simply $\sqrt{A^2 + B^2}$. It is easy to confirm for yourself that the corresponding rule applies in 3D, i.e. that a point (A, B, C) in the Cartesian system is just a distance $\sqrt{A^2 + B^2 + C^2}$ from the origin.

We will use a special notation for the magnitude of a vector: we write the normal symbol for the vector but place two vertical lines either side of it. So for the vector in our figure we can write

$$|\mathbf{v}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

This is simple to calculate in the Cartesian picture. Remember to try to simplify the result, so that if you obtain $\sqrt{32}$ this is better written as $4\sqrt{2}$.

1.5 Unit vectors

If a vector has magnitude of exactly one, then it is called a unit vector. We often use a special symbol to highlight such a vector: typically it is written with a ‘hat’ on top like $\hat{\mathbf{v}}$. Sometimes we are given some vector and asked to find the unit vector that points in the same direction. This is simple, once we use the following fact:

Vectors can be scaled (bigger/smaller) by multiplying by a factor.

Thus we can write $2\mathbf{v} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$, or instead shrink it with $\frac{1}{2}\mathbf{v} = \begin{pmatrix} 1 \\ \frac{3}{2} \\ 2 \end{pmatrix}$.

But now it is obvious how to find the unit vector corresponding to $\mathbf{v}$. We just need to multiply by the right factor to scale it down. That factor is of course just the reciprocal of the magnitude. So,

$$\frac{1}{|\mathbf{v}|}\mathbf{v} = \hat{\mathbf{v}} = \frac{1}{\sqrt{29}} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

As an aside, we should think about what it would mean to **multiply by a negative number**. In fact it is quite simple: if we multiply by (-1) then the new vector simply points in the opposite direction from the original.

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Obviously if we used some negative number other than (-1) , then we would be reversing the direction *and* scaling the vector as well.

1.5.1 The Cartesian unit vectors

Since we are using the Cartesian system to describe our vectors, there are three unit vectors that are especially helpful – namely, the ones that point out along the x , y and z axes. They just have special symbols, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. To see how useful these guys are, we need to look at how vectors can be added together.

1.6 Vector addition

What does it mean to add two vectors? Fortunately this operation has the simplest possible meaning: when we add two vectors, as in $\mathbf{a} + \mathbf{b}$, then we can simply picture vector $\mathbf{a}$ starting from some origin as usual, but vector $\mathbf{b}$ starting from the tip of vector $\mathbf{a}$. Then the vector from the origin to the end of vector $\mathbf{b}$ is their sum.

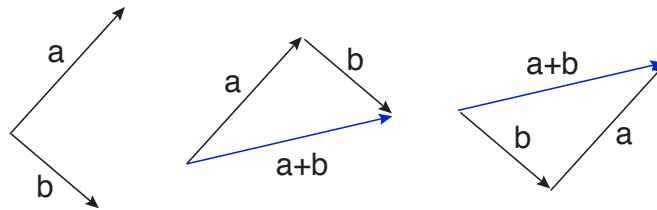


Figure 1.2: Summing together two vectors. We get the same result regardless of whether we follow $\mathbf{a}$ then $\mathbf{b}$, or instead take $\mathbf{b}$ first and then $\mathbf{a}$.

Thinking about things in this simple graphical way, we are able to see that it should not matter which order we add the vectors in. So,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad \text{and} \quad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

How about subtracting one vector from another, as in $\mathbf{a} - \mathbf{b}$? Well, remember that we can think of multiplying a vector by (-1) as re-orientating that vector in the opposite direction. So if we think of the subtraction as simply $\mathbf{a} + (-1)\mathbf{b}$ we can see this is like adding a vector opposite to $\mathbf{b}$ to the end of $\mathbf{a}$.

Now we can return to our Cartesian unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. Look again at the vector $\mathbf{v}$ in the earlier figure. We know that we can write this as $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

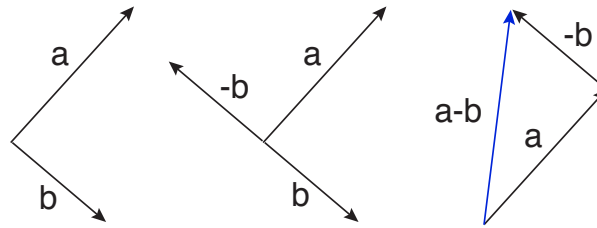


Figure 1.3: Subtracting one vector from another.

But having now thought about vector addition, we can also see that $\mathbf{v}$ is the sum of: a vector $2\hat{\mathbf{i}}$ along x , and vector $3\hat{\mathbf{j}}$ along y , and a vector $4\hat{\mathbf{j}}$ along z . Therefore

$$\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

Thus we have two equivalent ways of writing down a specific vector: either as the coordinates of the tip of the vector, or in terms of the sum of the three basic components. It's just a matter of convenience what you use in a particular situation.

If we need to add (or subtract) vectors that are specified in the Cartesian picture, then it's simply a matter of adding the components. So, if we were given a vector $\mathbf{u} = -3\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and asked to add this to the vector $\mathbf{v}$ given above, then we would have

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} -3 \\ \frac{1}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{7}{2} \\ 5 \end{pmatrix} = -\hat{\mathbf{i}} + \frac{7}{2}\hat{\mathbf{j}} + 5\hat{\mathbf{k}}.$$

1.6.1 Vector of zero magnitude

Sometimes when we are manipulating vectors, we find that the result is zero. An obvious example is when we subtract a vector from itself, $\mathbf{a} - \mathbf{a}$. Strictly speaking we should write the result as a vector zero, either as $\mathbf{0}$ or $\underline{0}$.

1.7 Obtaining the angles

Occasionally it is important to specify the direction of a vector in terms of its angles with respect to the x , y and z axes of the Cartesian frame, as shown in Fig. (1.4). This can look complex in a diagram, but it's actually easy. The key is that the angles, α , β and γ , are each part of a particular triangle involving a right angle. For example, for the γ angle, the triangle

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is formed by: the vector itself, the z -axis, and a line coming out from the z -axis at a right angle to that axis. Similarly with the x and y axes for α and β respectively.

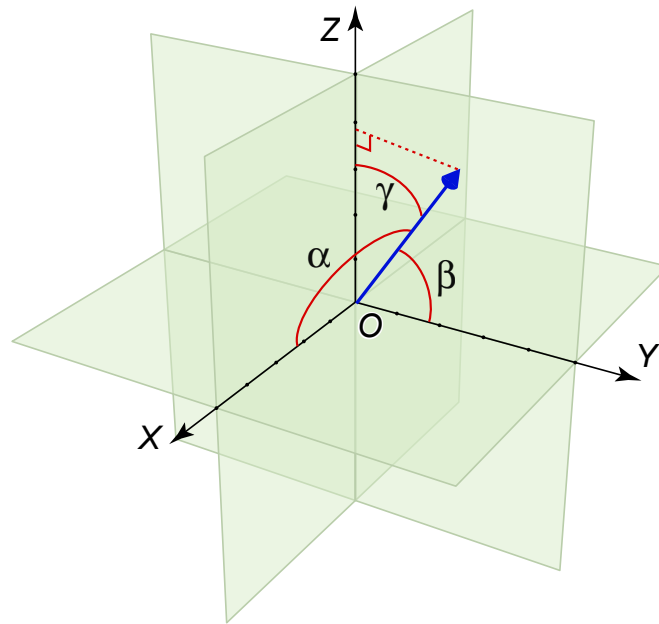


Figure 1.4: The angles between a vector and the Cartesian axes. The right angle triangle used to compute γ is indicated with the dashed line.

But in a right angle triangle, $\cos(\theta) = a/h$ relates a chosen angle θ , the length a of the side ‘adjacent’ and h the length of the hypotenuse. Using this we can quickly relate the components of a vector to its angles, because the length of the hypotenuse is just the magnitude of the vector, and length of the ‘adjacent’ edge of the triangle is just given by the x , y or z component of the vector (depending on whether you are after α , β or γ of course). To convince yourself of this it might help to compare Figs. (1.1) and (1.4).

Consider our earlier vector $\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and the x -axis, and suppose that we want to know the angle between the vector and the z -axis.

Describe the method here:

Generally, for some vector $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ we can write

$$\cos(\alpha) = \frac{v_x}{|\mathbf{v}|} \quad \cos(\beta) = \frac{v_y}{|\mathbf{v}|} \quad \cos(\gamma) = \frac{v_z}{|\mathbf{v}|}$$

These are called **the direction cosines**. We can see from the equations that $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$. Of course we should expect that these angles aren't independent: Otherwise it would take three free parameters merely to specify a direction; together with the magnitude of the vector, that would make four parameters just to specify a point in three-dimensional space!

1.8 Position vectors

One of the most useful aspects of vectors is that we can use them to describe locations, or the relative position of two locations. Often you may see a particular notation used when vectors are used in this rôle. If a position vector represents the location of some point, A , with respect to the Origin, then it may be written $\overrightarrow{\mathbf{OA}}$. Similarly, if we are going from point A on to point B , then we would write $\overrightarrow{\mathbf{AB}}$. As we know, going in the reverse direction is the same as multiplying by minus one, so

$$\overrightarrow{\mathbf{BA}} = -\overrightarrow{\mathbf{AB}}.$$

From the principle of vector addition we immediately see another equation

$$\begin{aligned} \overrightarrow{\mathbf{AB}} &= \overrightarrow{\mathbf{AO}} + \overrightarrow{\mathbf{OB}} \\ &= -\overrightarrow{\mathbf{OA}} + \overrightarrow{\mathbf{OB}} \end{aligned}$$

This last line is so useful that it is worth restating in the usual notation. If we have some vector $\mathbf{a}$ describing one point in space, and some other vector $\mathbf{b}$ describing a second point, then:

The vector $\mathbf{v}$ going from point $\mathbf{a}$ to point $\mathbf{b}$ is simply $\mathbf{v} = \mathbf{b} - \mathbf{a}$.

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It's important to remember this and get it the right way around! Let's do a simple example to illustrate:

Point A is located by vector $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and point B is located by $\mathbf{b} = \begin{pmatrix} 2 \\ \frac{2}{3} \\ -1 \end{pmatrix}$.

Equivalently we could use the notation $\overrightarrow{OA}$ and $\overrightarrow{OB}$ for these points. Now let's find the point that is exactly one quarter of the way from A to B .

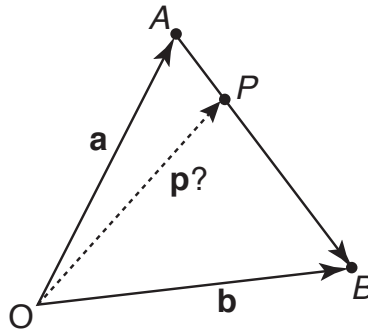


Figure 1.5: Find the location of P .

Solution:

1.9 Forces

Suppose that two, or more, forces are acting on an object. The rules for adding those forces in order to find the net total **resultant** force are exactly the rules of vector addition. Thus force is a vector quantity.

Importantly, if an object is stationary (or moving steadily without any acceleration) then the total force must be $\mathbf{0}$. Note, that is vector zero, meaning that the total force in the x , y and z direction is zero: three separate conditions (for a three-dimensional problem).

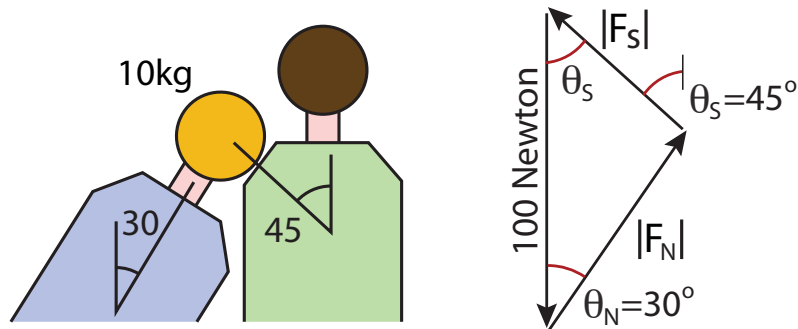


Figure 1.6: Zzzzz.

Let's take a quick example. Suppose that I am in a boring lecture and I fall asleep. I slump to one side so that my head is partly supported by my neck, at 30° to the vertical, and partly supported by the shoulder of the unfortunate woman next to me. My head rests there in such a way that the supporting force from her shoulder is at 45° to the vertical. My head has mass 10Kg and we'll say that the gravitational constant is $g = 10$ Newtons per Kg. What is the magnitude of the force on her shoulder?

Solution:

1.9.1 A short cut

There is a result from basic geometry that is helpful in working out problems where exactly three forces are involved, forming a triangle. It's called the **sine rule**. Importantly, the triangle does NOT have to be a right angle triangle. The rule states that the ratio of the length of a side to the sine of the angle opposite, is a constant.

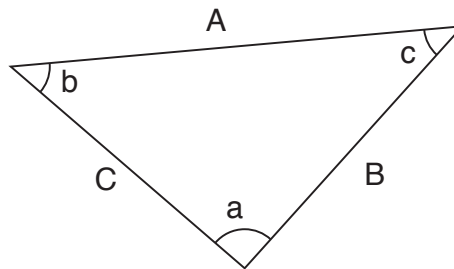


Figure 1.7: The sine rule. Note angle a is opposite to side A and so on.

$$\frac{A}{\sin(a)} = \frac{B}{\sin(b)} = \frac{C}{\sin(c)}$$

This can provide a short cut to solving problems. In the example above, we would note that

$$F_S = \frac{100}{\sin(180^\circ - \theta_S - \theta_N)} \sin(\theta_N) = \frac{100}{\sin(\theta_S + \theta_N)} \sin(\theta_N) \approx 51.8$$

Thus we find the same answer. Of course this means that the trigonometric expression must have been the same – check this as an exercise.

1.10 Equations for lines and planes

1.10.1 Lines

When we looked at position vectors, we were using a vector to specify a particular point in space. Here we'll see how, if we introduce a freely varying scalar variable into a vector, we can describe a whole line of points.

The simplest example is this:

$$\mathbf{r} = \lambda \hat{\mathbf{i}}$$

where λ is a number that's free to take any value. Think about what this means. As we freely adjust λ , the magnitude of the vector $\mathbf{r}$ will increase, or decrease, along the x axis. When we choose a negative value for λ the vector will invert (so that the vector points back along the negative x axis). So, by allowing λ to be free we are really encompassing the entire set of points that lie along the x axis. As an extension of this idea, we could write any line that passes through the origin as $\mathbf{r} = \lambda \mathbf{v}$, where vector $\mathbf{v}$ is some fixed vector for the position of any point along our line.

Now consider this:

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}$$

What's changed is that we have a fixed vector $\mathbf{p}$ added to a freely scaling vector, $\lambda \mathbf{v}$. The point $\mathbf{r} = \mathbf{p}$ must lie on the line (because that is obtained by setting $\lambda = 0$). Since any straight line is defined completely if we specify the direction together with one point that it passes through, it follows that this is a general equation for a straight line in vector notation.

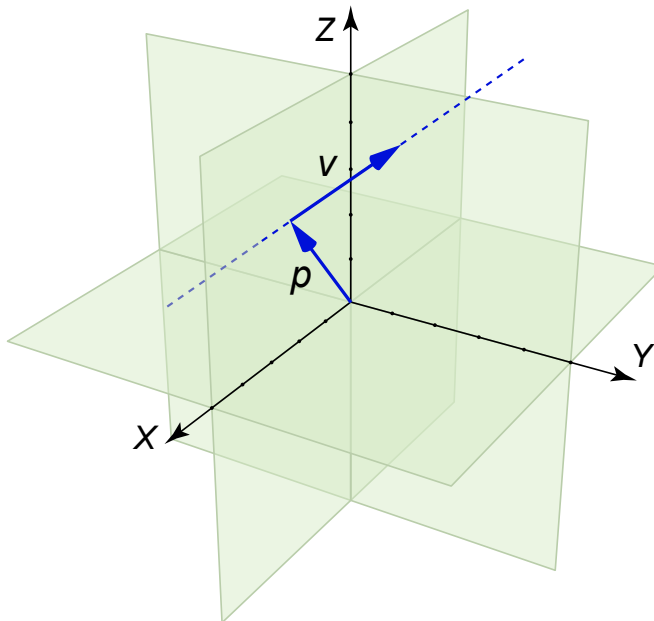


Figure 1.8: Illustration of the vector equation for a line, $\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}$.

Under what conditions could the equation $\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}$ describe a line that passes through the origin?

Non-uniqueness. It is important to note that there is not just one vector equation for a given line. Because the job of vector $\mathbf{p}$ is just to get us from the origin to **some place** along the line, clearly there are an infinity of choices for $\mathbf{p}$. Similarly, because $\mathbf{v}$, the vector parallel to the line, is multiplied by the free variable λ , it follows that the magnitude of $\mathbf{v}$ does not matter.

1.10.2 Planes

Having seen how to write the vector equation for a straight line, it is a simple step further to see how to write the equation for a plane. By varying one free parameter in our vector, we swept out our line. Suppose we add in a second freely scalable vector; a simple example would be

$$\mathbf{r} = \lambda \hat{\mathbf{i}} + \mu \hat{\mathbf{j}}$$

We could put $\mu = 0$ and then just vary λ as before, to sweep through all the points along the x -axis. But if we pause and vary μ instead, then we will move out from wherever we are along that x -axis and travel parallel to the y -axis. By varying both free parameters we can cover the whole x - y plane.

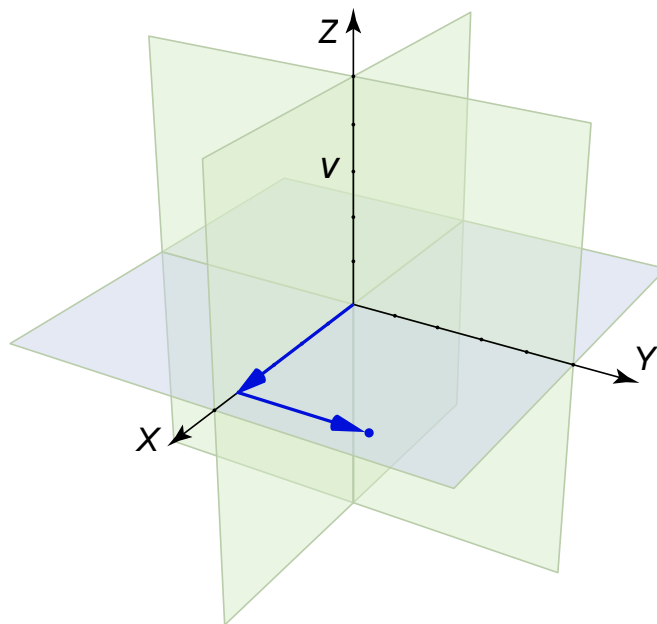


Figure 1.9: Illustration of the simple vector equation for a plane, $\mathbf{r} = \lambda \hat{\mathbf{i}} + \mu \hat{\mathbf{j}}$.

To generalise this to any plane, we just replace $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ with any two **non-parallel** vectors, and we add an offset vector $\mathbf{p}$ to shift the plane away

from the origin.

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}_1 + \mu \mathbf{v}_2$$

As with the line equation earlier, the vector equation for a plane is not unique. Presently we will encounter a second, quite different way of defining a plane using vectors; that other method will allow us to translate to the Cartesian plane equation.

1.10.3 Solving for intersections

Suppose that we are told that two lines intersect, or that a line intersects a plane. How will we find the point of intersection? Let's start with a simple case. Consider the line equation

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{v} \quad \text{where} \quad \mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}.$$

And suppose we are asked to find where this intersects with the x - y plane (for which the vector equation was written above).

All we need do is notice that any point in the x - y plane should be expressed purely in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, without any component of $\hat{\mathbf{k}}$. So, for what value of λ will our vector $\mathbf{r}$ have zero $\hat{\mathbf{k}}$ component? There is $3\hat{\mathbf{k}}$ coming from $\mathbf{p}$ and $2\lambda\hat{\mathbf{k}}$ from the $\lambda\mathbf{v}$ part. To make these cancel we must choose $\lambda = -\frac{3}{2}$. Then the point of intersection is

$$\mathbf{r}_{xy} = \mathbf{p} - \frac{3}{2}\mathbf{v} = \begin{pmatrix} -\frac{7}{2} \\ 5 \\ 0 \end{pmatrix}.$$

That one was easy because we spotted that one of the three components had to vanish. More generally, we must deal with lines and planes which do not conveniently align with the basic axes and planes of the Cartesian system. Then it's a bit more work.

Consider problem of finding the intersection of the following two lines,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{p}_1 + \lambda \mathbf{v}_1 & \text{where} & \quad \mathbf{p}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} & \text{and} \\ \mathbf{r}_2 &= \mathbf{p}_2 + \mu \mathbf{v}_2 & \text{where} & \quad \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}. \end{aligned}$$

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Notice that it is important to use different symbols for the vectors, and crucially the free parameters, in each equation.

Solve the problem in the following space.

As an further exercise, solve for μ instead and check that the same point is then given by the $\mathbf{r}_2$ equation. Also check that λ and μ satisfy all three of our constraining equations.

When we need to find the intersection of a **plane** with a **line**, it's exactly the same process of writing down the three constraints and solving them. But this time there will be three free variables (one from the line, two from the plane), so there is slightly more work to do!