

Vectors, Matrices and Determinants

Simon C Benjamin

Section 1 Continued (bundle 2 of 2)

Here we'll introduce two important ways of combining vectors: the dot product, the cross product. For each case we'll see how various new applications are enabled.

1.11 Dot Product ('Scalar product')

The dot product, also called the scalar product, is a simple way to combine two vectors in order to create a single scalar number (which may be positive or negative). For our purposes we will define the dot product as

$$\mathbf{a} \cdot \mathbf{b} \equiv |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

where the θ is understood to be the angle between the two vectors. Note that since $\cos(\theta) = \cos(-\theta)$ it follows that we don't need to specify which of the vectors we measure the angle **from** and in fact

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \quad (1.1)$$

However it is important to measure the angle **between** the vectors. If the angle is $> 90^\circ$ then the dot product will be a negative quantity.

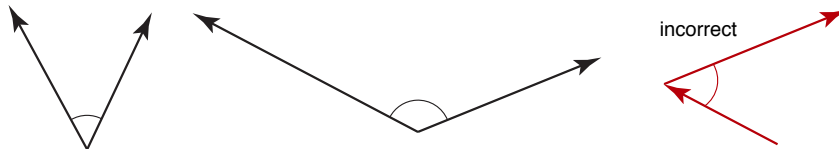


Figure 1.1: Left: dot product positive. Middle: negative. Right: incorrect.

If we take the dot product of any vector with itself, we see that

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}| \cos(0) = |\mathbf{a}|^2$$

and sometimes this may be written as a^2 or $\mathbf{a}^2$ – note that if a vector is simply squared like this, it always means the dot product.

Now if we think about the Cartesian unit vectors, we see that

$$\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}||\mathbf{i}| \cos(0^\circ) = 1, \quad \mathbf{i} \cdot \mathbf{j} = |\mathbf{i}||\mathbf{j}| \cos(90^\circ) = 0, \quad \mathbf{i} \cdot \mathbf{k} = |\mathbf{i}||\mathbf{k}| \cos(90^\circ) = 0.$$

Thus in fact

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

and all other combinations yield zero because they are **orthogonal**.

It is helpful to think of $\mathbf{a} \cdot \mathbf{b}$ as the projection of $\mathbf{a}$ onto the direction of $\mathbf{b}$, multiplied by $|\mathbf{b}|$. When we look at it this way, it is straightforward to see from a graphical construction that

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

and of course it also follows from Eqn. (1.1) that we can multiply through the other way, for example $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$. Then we can conclude that

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$$

This associativity allows us to reach an important alternative expression for the dot product, when we consider two vectors each expressed in terms of the three Cartesian unit vectors.

$$\mathbf{a} \cdot \mathbf{b} = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \cdot (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) = a_x b_x + a_y b_y + a_z b_z$$

because all of the six other terms are orthogonal and therefore vanish. Of course this means that our earlier definition of the magnitude of a vector correctly fits:

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_x^2 + a_y^2 + a_z^2$$

and we see that

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2}}.$$

Examples

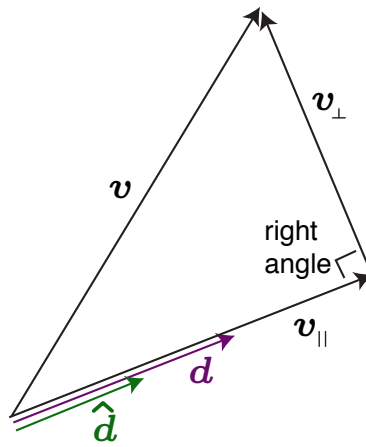


Figure 1.2: Resolving a vector.

1.12 Applications of the dot product

1.12.1 Resolving a vector: parallel and perpendicular components

Consider two vectors, $\mathbf{v}$ and $\mathbf{d}$, which are not parallel. Suppose that we want to express $\mathbf{v}$ as the sum of two particular vectors: one that is parallel to $\mathbf{d}$ and one that is perpendicular to it.

We can write

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \quad (1.2)$$

where we define $\mathbf{v}_{\parallel}$ to lie in the direction of $\mathbf{d}$, and we define $\mathbf{v}_{\perp}$ to be perpendicular to $\mathbf{d}$. We can capture these two facts by writing

$$\mathbf{v}_{\parallel} = l\hat{\mathbf{d}} \quad \text{and} \quad \mathbf{d} \cdot \mathbf{v}_{\perp} = 0. \quad (1.3)$$

where $\hat{\mathbf{d}}$ is the unit vector in direction $\mathbf{d}$, and l is some scalar that we'd like to find (note l could be negative – what would that mean?).

Let's find expressions for $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$ in terms of just $\mathbf{v}$ and $\mathbf{d}$. First we use the dot product property to get l . We'll evaluate $\hat{\mathbf{d}} \cdot \mathbf{v}$ by substituting in eqn.(1.2) and then using eqn.(1.3)

$$\hat{\mathbf{d}} \cdot \mathbf{v} = \hat{\mathbf{d}} \cdot (\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}) = \hat{\mathbf{d}} \cdot (l\hat{\mathbf{d}}) + 0 = l \quad (\text{since } \hat{\mathbf{d}} \cdot \hat{\mathbf{d}} = 1)$$

$$\text{Let's highlight this: } l = \hat{\mathbf{d}} \cdot \mathbf{v} \quad \text{and thus} \quad \mathbf{v}_{\parallel} = (\hat{\mathbf{d}} \cdot \mathbf{v})\hat{\mathbf{d}} \quad (1.4)$$

In words: To 'resolve' a vector into a certain direction, the key thing is to perform a dot product with the *unit* vector in that direction.

Finally we can get $\mathbf{v}_\perp$ simply by rearranging the definition:

$$\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel = \mathbf{v} - (\hat{\mathbf{d}} \cdot \mathbf{v})\hat{\mathbf{d}} \quad (1.5)$$

And of course we can check that this is indeed perpendicular with **another** use of dot product:

$$\hat{\mathbf{d}} \cdot \mathbf{v}_\perp = \hat{\mathbf{d}} \cdot \mathbf{v} - (\hat{\mathbf{d}} \cdot \mathbf{v})\hat{\mathbf{d}} \cdot \hat{\mathbf{d}} = 0.$$

Example:

1.12.2 Alternative equation for a plane

The dot product gives us a new and very elegant way to write the equation for a plane. Recall that previously we worked with

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}_1 + \mu \mathbf{v}_2$$

where $\mathbf{p}$ was the vector from the origin to any point in the plane, while $\mathbf{v}_1$ and $\mathbf{v}_2$ were vectors representing displacements within the plane. Now suppose that we are given $\mathbf{p}$ as before, but rather than $\mathbf{v}_1$ and $\mathbf{v}_2$ we are instead given

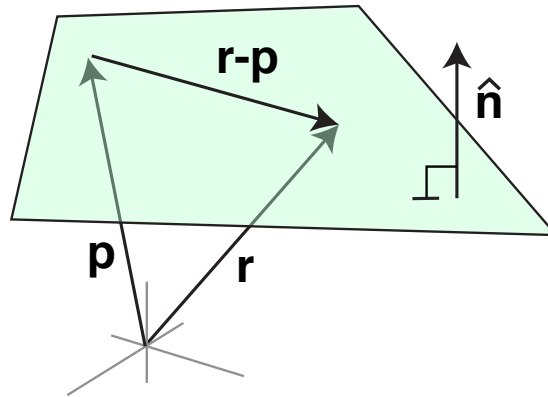


Figure 1.3: The alternative way to define a plane.

$\hat{n}$, the unit vector that is normal to the plane. (Later, when we introduce the cross product, we'll see how to get $\hat{n}$ if we are given $\mathbf{v}_1$ and $\mathbf{v}_2$).

Consider some general vector $\mathbf{r}$, how can we tell if it is the position vector of a point in the plane? Think about $\mathbf{r} - \mathbf{p}$, the vector between $\mathbf{r}$ and our known point $\mathbf{p}$. If indeed $\mathbf{r}$ is in the plane, then the vector between them is a displacement **within** the plane and therefore it must be perpendicular to the normal vector $\mathbf{n}$. That is,

$$\hat{n} \cdot (\mathbf{r} - \mathbf{p}) = 0.$$

Moreover, any $\mathbf{r}$ that satisfies this must be the position vector of a point in the plane. So in fact this constitutes an equation for the plane! Note that we have not needed to use the 'free parameters' λ and μ at all. In fact since both $\hat{n}$ and $\mathbf{p}$ are constants, and therefore $\hat{n} \cdot \mathbf{p}$ is just some constant, we can also write the equation as

$$\hat{n} \cdot \mathbf{r} = c.$$

where of course $c = \hat{n} \cdot \mathbf{p}$. Actually c is an interesting distance too – **what?**

One final thing to note: this new form of vector equation for a plane $\hat{n} \cdot (\mathbf{r} - \mathbf{p}) = 0$ is very easy to translate to/from a usual Cartesian form as an equation constraining x , y and z coordinates. We just write $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{p} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ and then we have

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0.$$

Example:

1.12.3 Angle between two planes

Suppose that we are thinking about two different planes, defined by equations $\hat{n}_1 \cdot \mathbf{r} = c_1$ and $\hat{n}_2 \cdot \mathbf{r} = c_2$. Assuming that the planes are not parallel, then they will intersect with one another. We might ask, what is the angle between these planes?

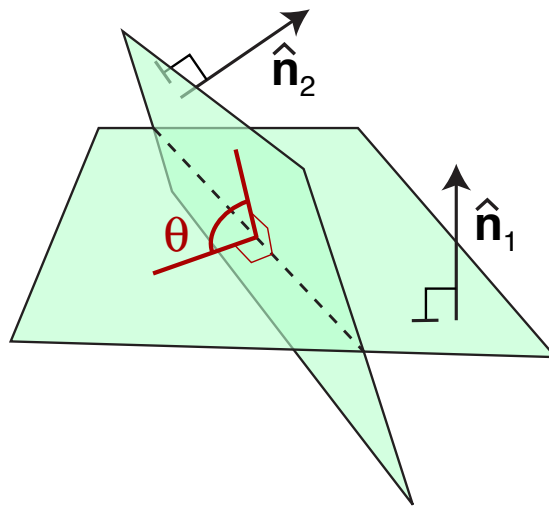


Figure 1.4: Intersecting planes. Easier than it looks!

Actually we can get different angles depending on how we measure – but the largest angle is when we measure at right angles to the line of intersection (see Fig) and this is what is meant by “the angle between two planes”. How

can we get that angle? Fortunately it is very easy: it's the same as the angle between the normal vectors of the two planes. If we've scaled our normal vectors to unit length then it's very easy indeed; the angle θ between planes is given by

$$\cos(\theta) = \hat{n}_1 \cdot \hat{n}_2$$

1.12.4 Distance from point to plane

Using very similar ideas to those above, we can write down the distance from some point $\mathbf{a}$ to a plane, if that plane is given by specifying the normal $\hat{n}$ and an arbitrary point in the plane, $\mathbf{p}$.

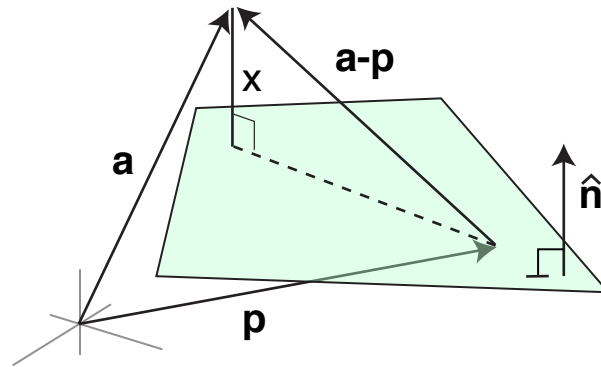


Figure 1.5: The distance x of a point to a plane.

We are looking for the distance x shown in the Fig. (1.5). Consider the vector along the hypotenuse, i.e. the vector $\mathbf{v} = \mathbf{a} - \mathbf{p}$. If we can find the component of this vector that is parallel to the normal vector $\hat{n}$, then the desired distance (x in the Figure) is just the magnitude. But we have already done this kind of thing – it's the problem of finding $v_{\parallel}$ in sub-section 1.12.1. The only difference is now we are interested in $\hat{n}$ instead of $\hat{\mathbf{d}}$. So,

$$\text{distance} = \hat{n} \cdot \mathbf{v} = \hat{n} \cdot (\mathbf{a} - \mathbf{p})$$

(Strictly we should put $||$ around this expression, since the dot product can be a negative but we are interested in the length).

How about if we were given the equation of the plane in the form $\hat{n} \cdot \mathbf{r} = c$? Then we would not have the reference point $\mathbf{p}$ in the plane. But we know from the previous section that $c = \hat{n} \cdot \mathbf{p}$, so we can still write down the required distance:

$$\text{distance} = |\hat{n} \cdot (\mathbf{a} - \mathbf{p})| = |\hat{n} \cdot \mathbf{a} - c|$$

Example:

1.12.5 Equation for a sphere

Consider a sphere, centred at some position $\mathbf{p}$ and having radius s . Immediately we can write down an equation for any point $\mathbf{r}$ on the surface of the sphere, just by thinking about the vector $\mathbf{r} - \mathbf{p}$ that runs from the centre of the sphere to the surface:

$$(\mathbf{r} - \mathbf{p}) \cdot (\mathbf{r} - \mathbf{p}) = s^2$$

A somewhat challenging task is to solve for the intersection of a line with the sphere – try it!

1.12.6 Obtaining trig identities: cosine formula

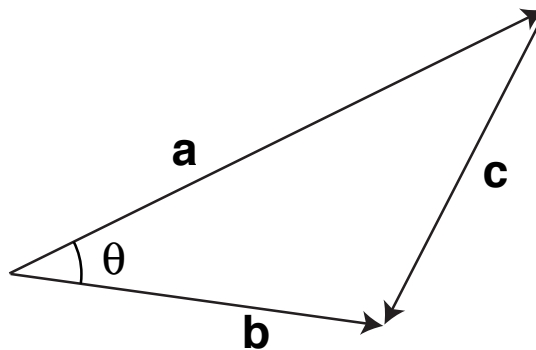


Figure 1.6: The cosine rule.

The dot product can allow us to quickly obtain the so-called cosine formula. Consider some triangle with edges corresponding to vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. We can immediately write

$$\begin{aligned}\mathbf{c} \cdot \mathbf{c} &= |\mathbf{c}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)\end{aligned}$$

That is,

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

1.12.7 Force and work done

We noted earlier that force is a vector quantity; for example when several forces act on an object, their sum should be $\mathbf{0}$ if the object is to remain static (or moving with constant velocity). Now that we can use the dot product, we are able to discuss the **work** done by a force. In physics, the work done is a quantity of energy depending on (a) the magnitude of the force, and (b) the distance moved **in the direction of the force**. Therefore,

$$\text{work done} = \mathbf{F} \cdot \mathbf{d}$$

when force $\mathbf{F}$ acts, and the object moves through a displacement $\mathbf{d}$. The most simple case is when $\mathbf{F}$ and $\mathbf{d}$ align; in that case, the work done is just Fd . However, if the force is perpendicular to the direction of motion, no work is done. What is an example of this situation?

Example

1.13 Cross product (or Vector product)

The dot product was an operation that yielded a scalar from two vectors. We will now look at the **cross product** operation, which generates a new vector from two original vectors.

For our definition, we will say that the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \times \mathbf{b} \equiv |\mathbf{a}||\mathbf{b}| \sin(\theta) \hat{v}$$

where $\hat{v}$ is a unit vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and θ is the small angle between $\mathbf{a}$ and $\mathbf{b}$, so that $0 \leq \theta \leq 180^\circ$. Now we also need to specify the **sense** of $\hat{v}$, i.e. which of the two possible ways it points along the perpendicular direction. The sense is defined by the so-called right-handed screw rule: We imagine grasping the line vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. If the fingers wrap around in the direction of the angle from $\mathbf{a}$ to $\mathbf{b}$, then the thumb points in the direction of $\hat{v}$!

We see that if we try to compute the cross product of a vector with itself, e.g. $\mathbf{a} \times \mathbf{a}$, then the result must be $\mathbf{0}$ since the angle in question will then be $\theta = 0$.

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

From the definition, we can also see that swapping the role of $\mathbf{a}$ and $\mathbf{b}$ would reverse the direction of $\hat{v}$. Therefore,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

Formally we say that the cross product operation is not commutative. (Compare to the dot product, which is commutative). However, the cross product is still distributive, meaning that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

(we will not prove this as the proof is long). As we did with the dot product, let's write the possible combinations of the Cartesian unit vectors. For starters, all the combinations of the vectors with themselves are zero:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

Now when we do the cross product of any two different Cartesian unit vectors, the new vector must simply point in the direction of the third vector! (Because the three together are all orthogonal to each other – so if we ask for a vector that is perpendicular to both $\mathbf{i}$ and $\mathbf{j}$ then that must be in the direction of $\mathbf{k}$). Also the magnitude of the new vector must be one, because

both the originals are unit vectors. Therefore, all we need to do is decide in each case whether we need a minus sign. In fact when we use the right hand screw rule to check, we find that,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Then of course, since generally $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, we know that

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

The easy way to remember this is by thinking of writing i , j and k in a circle. If we go clockwise (called ‘cyclic’), the result is positive; if anticlockwise (‘anti-cyclic’), then negative.

Now we are in a position to write out the full expression for the cross product of two vectors, when each is initially expressed in terms of the components $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \times (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \\ &= a_x b_y \mathbf{k} - a_x b_z \mathbf{j} - a_y b_x \mathbf{k} + a_y b_z \mathbf{i} + a_z b_x \mathbf{j} - a_z b_y \mathbf{i} \\ &= (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k} \end{aligned} \quad (1.6)$$

Using this explicit expression, it is possible to confirm that the new vector is indeed perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (just by taking the dot products and checking that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$).

Now although the expression for the cross product looks complex, there is a simple pattern there. Each term, $\mathbf{i}$, $\mathbf{j}$ or $\mathbf{k}$, is multiplied by a factor made from the ‘other’ a and b coefficients, with the signs following the cyclic/anti-cyclic rule.

Some people just remember this rule, or have some other way to remember the pattern in Eq. (1.6). Other people find it helpful to work out the cross product using a particular construction that is really an instance of a **determinant**, something that we will meet properly later in the course. **You should feel free to memorise any procedure for working out the cross product that happens to work for you!** In the following we’ll look at the determinant option:

We write a three-by-three matrix of numbers, between two vertical lines. Along the top we write $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. On the middle row we write a_x , a_y and a_z . And on the bottom row we write b_x , b_y and b_z .

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Now to work out the right coefficient of $\mathbf{i}$, we cover both the row and column containing $\mathbf{i}$, to leave a two-by-two square. Then we multiply the terms on the leading diagonal and subtract the product of the off-diagonal terms. The same goes for the coefficient of the $\mathbf{j}$ and $\mathbf{k}$ components, except for $\mathbf{j}$ we have to remember an overall minus sign (because of the cyclic rule).

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \mathbf{k} \\ &= (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}\end{aligned}$$

Example

1.13.1 Area of triangle and of parallelogram

Let $\mathbf{a}$ and $\mathbf{b}$ be two sides of a triangle. We can find the area of the triangle by the following argument, involving dividing the triangle into two right angle triangles as shown in the Figure.

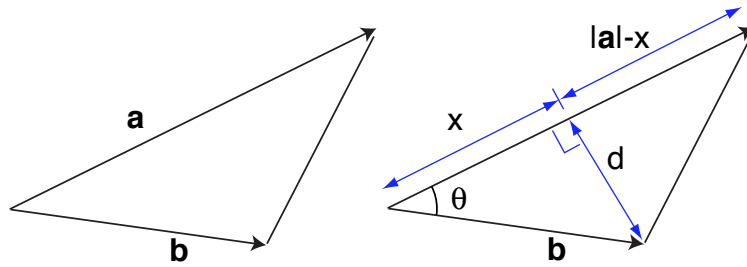


Figure 1.7: The area of a triangle.

$$\begin{aligned}
 \text{Area of triangle} &= \frac{1}{2}xd + \frac{1}{2}(a - x)d \\
 &= \frac{1}{2}xb \sin(\theta) + \frac{1}{2}(a - x)b \sin(\theta) \\
 &= \frac{1}{2}ab \sin(\theta) \\
 &= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|
 \end{aligned}$$

Consider a parallelogram having two adjacent sides $\mathbf{a}$ and $\mathbf{b}$. The area is just twice that of the corresponding triangle, we also see that

$$\text{Area of parallelogram} = |\mathbf{a} \times \mathbf{b}|$$

Example

1.13.2 Distance of a point from a straight line

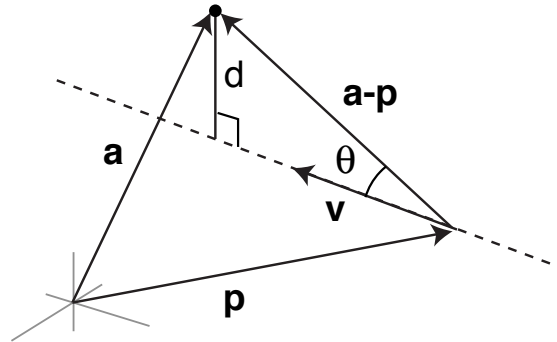


Figure 1.8: The distance of a point to a line.

Suppose we have been given the equation for a line, $\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}$, and some point $\mathbf{a}$. We want to know the shortest distance from $\mathbf{a}$ to the line. Looking at the construction in the figure, we see that the answer must be $|(\mathbf{a} - \mathbf{p})| \sin(\theta)$. But this is just

$$\text{distance} = |(\mathbf{a} - \mathbf{p}) \times \hat{\mathbf{v}}| = \frac{1}{|\mathbf{v}|} |(\mathbf{a} - \mathbf{p}) \times \mathbf{v}|$$

1.13.3 The vector normal to a plane

Earlier we played with two different kinds of equation for a plane. One form was line

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}_1 + \mu \mathbf{v}_2$$

while another looked quite different

$$\hat{\mathbf{n}} \cdot \mathbf{r} = c$$

where $c = \hat{\mathbf{n}} \cdot \mathbf{p}$. Suppose we want to derive the second, more compact expression from the first. All we need to do is to find the vector $\hat{\mathbf{n}}$ normal to a plane using the two vectors that lie within that plane. But then of course the normal vector $\hat{\mathbf{n}}$ is precisely the vector that is simultaneously perpendicular to both vectors in the plane. And fortunately the cross product is exactly the operation we need in order to get that!

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 \quad \text{and} \quad \hat{\mathbf{n}} = \frac{1}{|\mathbf{n}|} \mathbf{n}$$

Example

1.13.4 Alternative line equation

It is worth quickly noting that the cross product allows us to write the line equation in a new way (it is perhaps less intuitive but it gets rid of the free parameter). Our previous method was to write

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{v}$$

therefore

$$\mathbf{r} - \mathbf{p} = \lambda \mathbf{v}$$

and so by taking the cross product with $\mathbf{v}$ on both sides, we find

$$(\mathbf{r} - \mathbf{p}) \times \mathbf{v} = \mathbf{0}$$

1.13.5 Torque

Further extending our ideas about forces, we can now discuss torque. This is a vector quantity that is important when we are interested in objects that rotate.

Suppose that some object is free to rotate about a fixed axis. Imagine a lever attached to the object, and a force applied to the end of the lever. Then the magnitude of the torque is simply the magnitude of the force applied, multiplied by the **perpendicular** distance from the axis. The meaning of this definition is best understood by looking at the Figure.

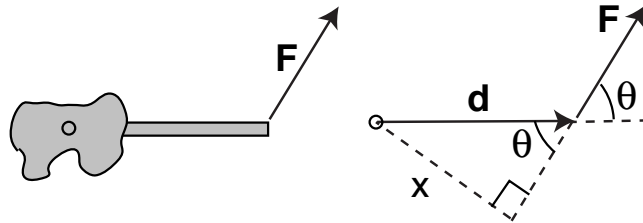


Figure 1.9: Finding the torque. The ‘perpendicular distance’ is x .

The expression for the torque is

$$\mathbf{T} = \mathbf{d} \times \mathbf{F}$$

which gives the correct magnitude as defined above, and a direction that is perpendicular to both $\mathbf{F}$ and $\mathbf{d}$. When several different torques act, we can sum them to find the net torque.

Example