

Vectors, Matrices and Determinants

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Chapter 5. Eigenvalues and Eigenvectors

Here we'll start by introducing coordinate transformations. Then we will go on to think about how to diagonalise a matrix, a challenge that will lead us to encounter eigenvectors and eigenvalues.

5.1 Rotation of coordinates

We've seen how an appropriate matrix can cause a rotation of any vector that it multiplies, and hence we pictured rotating a solid defined by the position vectors of its corners. An example was the matrix $\mathbf{R}_Z(\theta)$, defined as a rotation (in the anticlockwise direction) about the z -axis. We wrote

$$\mathbf{R}_Z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

But there is a second way to interpret this rotation matrix. We could say that the vectors are staying static but the **coordinate system** itself is rotating. Previously we have looked at rotations about the x , y and z axes. In general, how should we write a matrix $\mathbf{R}$ that transforms from some old

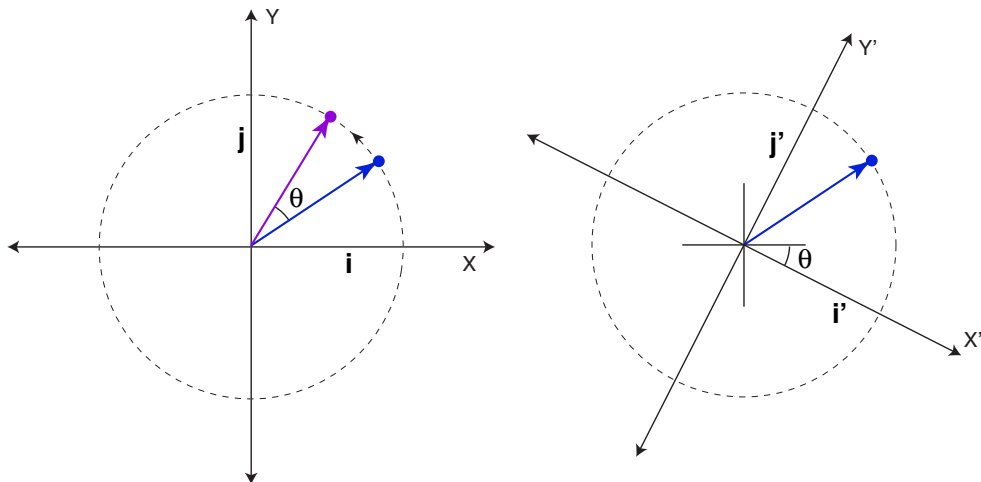


Figure 5.1: Rotation of a vector in a fixed coordinate frame, versus rotation of a coordinate frame while vector is static.

set of coordinates, call them x_1 , x_2 and x_3 , to some new set of coordinates x'_1 , x'_2 and x'_3 ? The key is to think of how a vector would be written in the old and new systems. For each coordinate system, there will be a set of three unit basis vectors (like $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ in the case of x , y and z). Then we can write,

$$\begin{aligned}\mathbf{v} &= v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3 \\ &= v'_1\hat{\mathbf{e}}'_1 + v'_2\hat{\mathbf{e}}'_2 + v'_3\hat{\mathbf{e}}'_3.\end{aligned}$$

Here the vectors $\hat{\mathbf{e}}_i$ are the unit vectors in the directions of the x_i axes, and similarly $\hat{\mathbf{e}}'_i$ are the unit vectors in the directions of the x'_i axes. We will assume that the old basis and the new basis each form an orthogonal set (i.e. $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ and $\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}'_j = \delta_{ij}$). Note that the vector $\mathbf{v}$ is not changing — we are simply writing it in two different coordinate frames.

We want the matrix $\mathbf{R}$ that will transform the components v_i into the new components v'_i . We notice that

$$\begin{aligned}v'_1 &= \hat{\mathbf{e}}'_1 \cdot \mathbf{v} \\ &= v_1\hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_3\end{aligned}$$

Similarly,

$$\begin{aligned}v'_2 &= v_1\hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_3 \\ v'_3 &= v_1\hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_3.\end{aligned}$$

Therefore we find our 3×3 rotation matrix $\mathbf{R}$ by writing,

$$\begin{pmatrix} v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \quad (5.1)$$

Note: this matrix satisfies the condition of orthogonality (how can we see that?). Rotation matrices are always orthogonal, and therefore we will be able to make use of the rule $\mathbf{R}^{-1} = \mathbf{R}^T$.

5.2 Tensor equations

We are familiar with using vectors in physical equations, when the quantities involved naturally have both direction and magnitude. One example we have seen was

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the force applied to an object, m is its mass, and $\mathbf{a}$ is the resulting acceleration. In this equation, the two vector quantities are related to one another by a constant m which is simply a scalar quantity. However, there are many equations which relate two vector quantities using a full matrix in place of the constant. An example you will meet in the question sheet is

$$\mathbf{J} = \boldsymbol{\sigma} \mathbf{E}.$$

Here $\mathbf{E}$ is an electric field applied to some substance, and $\mathbf{J}$ is the resulting current density. In a simple homogeneous substance, the current would simply be in the direction of the electric field, and so we would use a σ that is simply a scalar: the electrical conductivity. However, in more complex structures such as crystals, the direction of the current may differ from that of the applied field. Then in general we need to use a $\boldsymbol{\sigma}$ which is a matrix: the electrical conductivity tensor. Generally, whenever we have some property of a system that maps one vector quantity to another like this, we call it a **tensor quantity**.

As soon as we begin to think of $\boldsymbol{\sigma}$ as a property of the material, we can see that it must be referred to some coordinate system just as vectors are referred to a particular system. But then, if we rotate the coordinate system, then the elements of $\boldsymbol{\sigma}$ will have to change (just as some particular vector $\mathbf{v}$ would be described by a new set of elements if we rotate the coordinate frame). We know that for vectors,

$$\mathbf{v}' = \mathbf{R} \mathbf{v}$$

where $\mathbf{R}$ is our coordinate rotation. So what is the expression for the rotated tensor? We can figure this out as follows. Suppose that in the original coordinate system, we have some equation

$$\mathbf{a} = \mathbf{K} \mathbf{b}.$$

where $\mathbf{K}$ is our tensor quantity (electrical conductivity in the previous example), and $\mathbf{a}$ and $\mathbf{b}$ are two vector quantities (like current and electric field, above). Now rotating the coordinate system does not change the physical law, so we should write

$$\mathbf{a}' = \mathbf{K}' \mathbf{b}'$$

where the quantities with the prime symbol, $'$, are simply the same old quantities in the new coordinate frame. However, we know that

$$\mathbf{a}' = \mathbf{R} \mathbf{a} \quad \text{and} \quad \mathbf{b}' = \mathbf{R} \mathbf{b}.$$

Therefore we can write,

$$\mathbf{R} \mathbf{a} = \mathbf{K}' \mathbf{R} \mathbf{b}.$$

But then, we can just multiply from the left by the inverse $\mathbf{R}^{-1}$,

$$\begin{aligned} \mathbf{R}^{-1} \mathbf{R} \mathbf{a} &= \mathbf{R}^{-1} \mathbf{K}' \mathbf{R} \mathbf{b} \\ \Rightarrow \mathbf{a} &= \mathbf{R}^{-1} \mathbf{K}' \mathbf{R} \mathbf{b} \end{aligned}$$

since $\mathbf{R}^{-1} \mathbf{R}$ is simply the identity, which has no effect. But now, comparing this with our original equation

$$\mathbf{a} = \mathbf{K} \mathbf{b}$$

we see that

$$\mathbf{K} = \mathbf{R}^{-1} \mathbf{K}' \mathbf{R}$$

which is the same as

$$\mathbf{K}' = \mathbf{R} \mathbf{K} \mathbf{R}^{-1}.$$

So, we have found that when we must re-express some tensor quantity in a rotated frame, we need to multiply by $\mathbf{R}$ from the left and $\mathbf{R}^{-1}$ from the right.

In general, finding an inverse matrix is hard; but remember that for orthogonal matrices it is easy:

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

where $\mathbf{R}^T$ is simply the transpose. Since any matrix representing a rotation will indeed be orthogonal, we can simply write the tensor transformation as

$$\mathbf{K}' = \mathbf{R} \mathbf{K} \mathbf{R}^T.$$

Example : Find $\mathbf{K}'$ given $\mathbf{K} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

5.3 Diagonalisation; Eigenvectors and Eigenvalues

We saw from the above example a tensor may look complex in one coordinate system, possessing non-zero off-diagonal terms, but that same tensor may be diagonal if we can rotate to just the “right” coordinate system. Obviously, it is desirable to be able to find that ideal coordinate system if it exists. Expressing a tensor in this special coordinate system is called **diagonalisation**.

Here we will assume that such a coordinate system **does** exist, and develop a method for discovering it. Later we will return to the question of whether it is always possible.

Suppose that we are given some tensor $\mathbf{M}$ that is not diagonal. We want to find the rotation $\mathbf{R}$ such that

$$\mathbf{M}' = \mathbf{R} \mathbf{M} \mathbf{R}^T$$

is diagonal. But if indeed it is diagonal, then

$$\mathbf{M}' = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

for some a , b and c . So we can write

$$\mathbf{M}' \mathbf{v}' = \lambda \mathbf{v}' \quad \text{where} \quad (5.2)$$

$$\mathbf{v}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda = a \quad \text{or} \quad \mathbf{v}' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda = b \quad \text{or} \quad \mathbf{v}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda = c.$$

Here we see that the three special vectors $\mathbf{v}'$ are just the unit vectors of the new, rotated coordinate system that we want to find. Those unit vectors lie

along the so-called **principal axes** of the tensor. The diagonal elements a , b and c are called the **principal values** of the tensor.

Now, we know that there is some matrix $\mathbf{R}$ which relates the original coordinate frame and the new frame,

$$\mathbf{M}' = \mathbf{R} \mathbf{M} \mathbf{R}^T \quad \mathbf{v}' = \mathbf{R} \mathbf{v}$$

which means that Equation (5.2) becomes

$$(\mathbf{R} \mathbf{M} \mathbf{R}^T) (\mathbf{R} \mathbf{v}) = \lambda (\mathbf{R} \mathbf{v}).$$

Multiplying in from the left by $\mathbf{R}^T$ we exploit $\mathbf{R}^T \mathbf{R} = \mathbf{I}$, so that

$$(\mathbf{M} \mathbf{R}^T) (\mathbf{R} \mathbf{v}) = \lambda \mathbf{v}.$$

and finally we remember that matrix multiplication is associative, meaning that we can do the multiplication $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ in the middle of the expression. Thus,

$$\mathbf{M} \mathbf{v} = \lambda \mathbf{v}. \tag{5.3}$$

What does this equation tell us? It says that in the original frame there are some special vectors $\mathbf{v}$ which, when multiplied by $\mathbf{M}$, just pick up a scale factor. These scale factors λ are just the diagonal elements of the transformed $\mathbf{M}$. And the special vectors are precisely the ones that will become the principal axes when we rotate to the new frame. If we could find the three $\mathbf{v}$, we would have discovered the complete coordinate rotation (why?). The crucial question is, how can we find them.

This is called an **eigenvalue** problem; the constants λ in Equation (5.3) are the eigenvalues and the vectors $\mathbf{v}$ are the corresponding **eigenvectors**. To find them, first introduce an identity matrix $\mathbf{I}$,

$$\mathbf{M} \mathbf{v} = \lambda \mathbf{I} \mathbf{v}$$

so that we can write

$$(\mathbf{M} - \lambda \mathbf{I}) \mathbf{v} = \mathbf{0}.$$

Now, this is really a set of simultaneous equations, just like we looked at in Part 5 of these notes. It is of the form,

$$\mathbf{D} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where x , y , and z are the components of the vectors $\mathbf{v}$ that we want, and $\mathbf{D} = \mathbf{M} - \lambda \mathbf{I}$. Now, suppose that these equations are **independent**. Then

there is just one solution, and we can see it immediately: $x = y = z = 0$. But this is no good for us: we need to find three non-zero sets of values of x , y and z to give us our vectors. We recall that the condition for a set of equations to be dependent, is that $|\mathbf{D}| = 0$. Thus, we require

$$|\mathbf{M} - \mathbf{I}\lambda| = 0$$

and by solving this, we will find the possible λ . From there, we can find the corresponding vectors $\mathbf{v}$ easily. This is best illustrated with an example.

Example

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5.3.1 Normalising eigenvectors

In the above example, we first discovered the possible eigenvalues λ (also called principal values), and then we took each particular eigenvalue λ_i and worked out its corresponding eigenvector $\mathbf{v}_i$, by considering the equation

$$\mathbf{M} \mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

We found that this equation does not pin down the values of the components of $\mathbf{v}$ but rather specifies their ratios. This should always be the case when we find our eigenvectors. Thus we can choose to scale our eigenvectors to have unit magnitude, which is called **normalising** them. In other words, we add the condition that

$$\mathbf{v}_i \cdot \mathbf{v}_i = 1$$

in order to specify each $\mathbf{v}_i$ completely. (It is also worth remarking that the dot product between different eigenvectors is zero.)

Example

5.3.2 Finding the rotation matrix

We have seen how to find the eigenvalues and eigenvectors of a matrix. Recall that our motivation for doing this was to find out how to rotate a matrix into diagonal form. We wanted to find the ‘principal axes’, i.e. the best coordinate system, for describing some tensor quantity like conductivity. Ultimately we wanted the diagonal matrix $\mathbf{M}'$ in the expression,

$$\mathbf{M}' = \mathbf{R}\mathbf{M}\mathbf{R}^T$$

Now in fact, we have discovered that the eigenvalues of $\mathbf{M}$ already tell us the numbers down the diagonal of the rotated matrix $\mathbf{M}'$. And the corresponding eigenvectors tell us the principal axes (in the original coordinate frame). So basically we’ve found everything. Yet it is still useful to write down the rotation matrix itself. Fortunately, this is trivial! The matrix $\mathbf{R}$ is just composed by writing the eigenvectors as its rows. But crucially, they must be **normalised** for this to work.

$$\mathbf{R} = \left(\begin{pmatrix} \leftarrow & \hat{\mathbf{v}}_1 & \rightarrow \end{pmatrix} \\ \begin{pmatrix} \leftarrow & \hat{\mathbf{v}}_2 & \rightarrow \end{pmatrix} \\ \begin{pmatrix} \leftarrow & \hat{\mathbf{v}}_3 & \rightarrow \end{pmatrix} \right) \quad \mathbf{R}^T = \left(\begin{pmatrix} \uparrow \\ \hat{\mathbf{v}}_1 \\ \downarrow \end{pmatrix} \quad \begin{pmatrix} \uparrow \\ \hat{\mathbf{v}}_2 \\ \downarrow \end{pmatrix} \quad \begin{pmatrix} \uparrow \\ \hat{\mathbf{v}}_3 \\ \downarrow \end{pmatrix} \right). \quad (5.4)$$

We can see that this must be right, by comparing to our expression for a general rotation matrix, Equation (5.1). Why?

Example

5.3.3 Steps to diagonalisation

Let's write out the steps to diagonalise some matrix $\mathbf{M}$. Here $\mathbf{M}$ is an $n \times n$ matrix. We want the rotated form $\mathbf{M}' = \mathbf{RMR}^T$ which is diagonal.

- Solve the equation $|\mathbf{M} - \lambda\mathbf{I}| = 0$ to find the n values for the eigenvalues λ_i ($i = 1 \dots n$). These eigenvalues are in fact the numbers down the diagonal of $\mathbf{M}'$, and are also called the principal values.
- For each λ_i we need the corresponding the eigenvector $\mathbf{v}_i$. The way to find it is to put the specific λ_i into the equation $\mathbf{M}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ and so work out the elements of $\mathbf{v}_i$. Impose the condition $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ in order to **normalise** the eigenvectors.
- Having found the normalised eigenvectors, we can easily write down the rotation matrix $\mathbf{R}$ if we want, see Equation (5.4).

5.4 Symmetric tensors

In the examples we have seen, and those on the question sheet, the eigenvalues are all real numbers (not complex). Also, the eigenvectors formed an **orthonormal set**, that is to say, $\hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_j = \delta_{ij}$. In fact this will always prove to be the case if the original matrix, for which we are deriving eigenvalues and eigenvectors, is **symmetric**. Fortunately all the tensor quantities that you will meet do in fact have this property.

Aside: In fact these properties (real eigenvectors and orthogonal vectors) are associated with any **Hermitian** matrix: that is, a matrix that is equal to the complex conjugate of its own transpose. Since our matrices have only contained real numbers, performing the complex conjugate doesn't have any effect – then, the condition reduces to simply saying that a matrix is equal to its own transpose, i.e. it is symmetric. As a challenge, try to prove that a Hermitian matrix has real eigenvalues and orthogonal eigenvectors! Hint: it may be useful to know that $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$.

5.5 Invariants of a matrix

When we rotate a matrix to some new coordinate system, i.e. when we work out $\mathbf{M}' = \mathbf{RMR}^T$, then of course the elements of the new matrix $\mathbf{M}'$ differ from those of $\mathbf{M}$. However, there are certain combinations of the elements that do not change; they are called invariants of the matrix. In the following, we will consider a 3×3 matrix.

For a 3×3 matrix there are 3 invariants. The first invariant is simply the sum of the elements on the diagonal. This sum is called the **the trace of the matrix**. So, we can write

$$I_1 = \text{Trace}(\mathbf{M}) = M_{11} + M_{22} + M_{33}$$

and this will always be the same regardless of how we rotate $\mathbf{M}$.

A second invariant is the following:

$$I_2 = M_{11}M_{22} + M_{22}M_{33} + M_{33}M_{11} - M_{12}M_{21} - M_{23}M_{32} - M_{31}M_{13}.$$

Now in fact this invariant Q_2 is combined with the trace Q_1 , to produce an important invariant quantity that you may meet later in your studies: the von Mises shear stress.

Finally we note one further invariant:

$$\begin{aligned} I_3 = & M_{11}M_{22}M_{33} + M_{12}M_{23}M_{31} + M_{13}M_{21}M_{32} \\ & - M_{11}M_{23}M_{32} - M_{22}M_{13}M_{31} - M_{33}M_{12}M_{21} \end{aligned}$$

This is in fact simply the determinant of the matrix, $|\mathbf{M}|$. There is an exercise on the problem sheet, leading you to think about proving that these three quantities I_1 , I_2 and I_3 are indeed invariants.