Fourier Series, Fourier Transforms, and PDEs

Simon C Benjamin

Week 1: Introducing Fourier Series

Welcome to the lecture notes for this course! In this first set of notes, we will encounter Fourier Series, a powerful method for expressing periodic functions in terms of simple sine and cosine terms. But first, a reminder of some useful things.

1.1 Even, Odd and Periodic Functions

First, even and odd functions. They're quite self-explanatory. An **even function** gives the same value regardless of whether you change the sign of its argument:

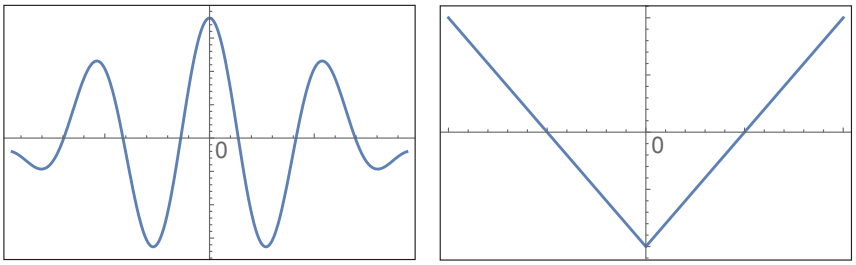

Figure 1.1: A function $f(x)$ is even if $f(x) = f(-x)$

Meanwhile, an **odd function** is just the opposite: if you change the sign of the value you feed in, i.e. $x \to -x$, then the function flips sign too.

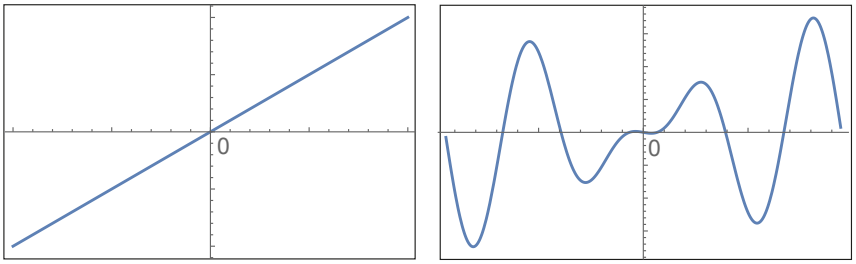

Figure 1.2: A function $f(x)$ is odd if $f(x) = -f(-x)$

It's obvious 'by inspection' that you'll get zero if you integrate an odd function between symmetric limits either side of zero, so $-K$ to K for example. But it takes a few lines to prove it!

Let $h(x)$ be an *odd* function, i.e. $h(-x) = -h(x)$. We can break up our symmetric integral:

$$\int_{-K}^{K} h(x)dx = \int_{-K}^{0} h(x)dx + \int_{0}^{K} h(x)dx$$

and then note the following...

$$\begin{aligned}
\int_{-K}^{0} h(x)dx &= -\int_{K}^{0} h(-y)dy \quad \text{using } y \equiv -x \\
&= \int_{K}^{0} h(y)dy \qquad \text{since } h \text{ is odd, i.e. } h(y) = -h(-y) \\
&= -\int_{0}^{K} h(y)dy \quad \text{reversing the limits of the integral.}
\end{aligned}$$

so we have

$$\int_{-K}^{K} h(x)dx = -\int_{0}^{K} h(y)dy + \int_{0}^{K} h(x)dx = 0$$

(the two terms are the same because x and y are just 'dummy variables' telling us how to do the integral; when we introduced y we gave it a definite relation to x but that doesn't matter here since x and y are dummies in two different terms.)

On to **periodic functions**. A function is periodic if it just repeats itself after a certain period (over and over again forever, and in both the directions of positive and negative x). If a function $f(x)$ is periodic with period P then we can write $f(x) = f(x + P)$.

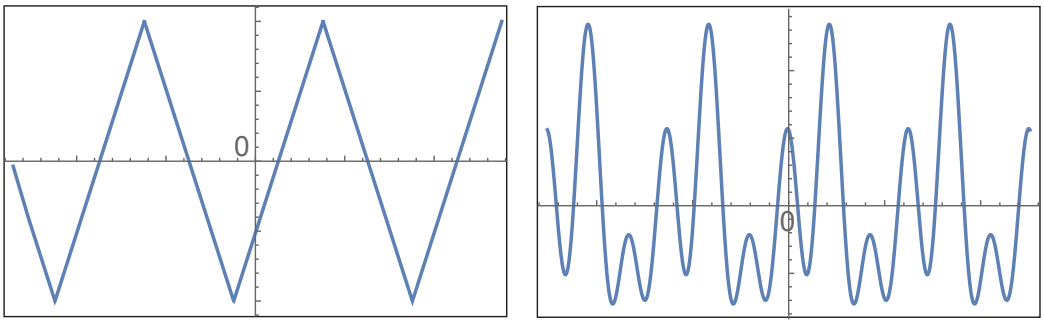

Figure 1.3: A function $f(x)$ is odd if $f(x) = f(x + P)$.

1.2 Adding together sine and cosine functions

We can see by playing that if we add together terms like $\sin(nx)$ and $\cos(nx)$ for various integer $n \geq 0$, we can get a pretty complex function. But since because we are using integer n we'll always have $f(x) = f(x + 2\pi)$.

So.... maybe we can deliberately put together a sum of $\sin(nx)$ and $\cos(nx)$ to try to make (as nearly as possible) some particular periodic function $f(x)$. The box below shows how we can use the tool matlab to visualise; another option is Mathematica – but are vastly capable maths systems.

Computer exercise:
Use matlab, Mathematica or something similar to experiment with constructing some complex functions by combining sine and cosine functions. (Example code below is for matlab) How can we make it non-periodic?

```
syms x;
f=2.5*cos(1.2*x)
          +0.9*sin(0.4*x);
fplot(f,[-30,30])
```

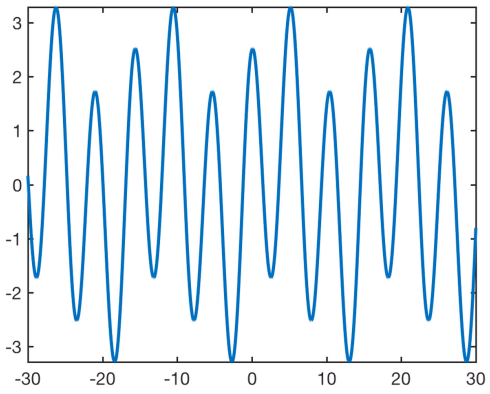

Figure 1.4: Example output.

1.3 Introducing the Fourier Series.

For some general, well behaved $f(x)$ which is periodic with a 2π period, which means $f(x) = f(x + 2\pi)$, let's *suppose* that we can write:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Warning! A few textbooks write the constant a_0 without the factor of $\frac{1}{2}$.

(Fortunately most books use the $a_0/2$ convention we are adopting, because it yields more compact expressions later on).

Then we ask, *how can we find out the right a_n and b_n to use for some particular $f(x)$*? We could try trial and error, maybe on a computer, until the resulting series looks right for the $f(x)$ we want. But obviously it is better to find expressions for a_n and b_n through a bit of mathematical manipulation!

1.4 Finding the a_n and b_n coefficients

Let's start by seeing if we can *get at* the value of a_0. The special thing about a_0 is that it is a constant term, whereas all the other a_n and b_n are in front of oscillatory terms.

In fact every one of those oscillatory terms, being sines and cosines with period $2\pi/n$, will all integrate to ZERO over one full period of $f(x)$, i.e. from 0 to 2π (or $-\pi$ to π or any other period). So: let's do that - integrate over one period to 'kill' all the terms except a_0:

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right) dx$$

so we find

$$\begin{aligned} \int_0^{2\pi} f(x)dx &= \frac{a_0}{2} \int_0^{2\pi} dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)dx + b_n \int_0^{2\pi} \sin(nx)dx \\ &= \frac{a_0}{2}\, 2\pi + 0 + 0 \end{aligned}$$

rearranging, we see we've managed to get at a_0 as we hoped!

$$\boxed{a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)dx}$$

But, can some similar trick allow us to get at the other constants, such as a_n for $n = 32$ for example? Somehow, we'd want to do something so that a term associated with the constant we are after, call it a_m, does *not* integrate to zero, while all the others including the fixed term a_0 do integrate to zero.

One way to make sure that the $\cos(mx)$ term (for example) does not integrate to zero is to square it - then it has to be positive everywhere. So

suppose we multiply all of $f(x)$ by $\cos(mx)$:

$$\begin{aligned}\cos(mx)f(x) &= \cos(mx)\left(\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nx)+b_n\sin(nx)\right)\\ &= \frac{a_0}{2}\cos(mx)+\sum_{n=1}^{\infty} a_n\cos(mx)\cos(nx)+b_n\cos(mx)\sin(nx)\end{aligned}$$

Now, in the integral $\int_0^{2\pi}\cos(mx)f(x)dx$, the a_m term that we are after won't integrate to zero, just as we wanted. But what about the other terms - will they disappear?

Well we can see that the a_0 term is going to integrate to zero. For the others, we have to work them out. This is an exercise in the problems. We'll find that:

$$\begin{aligned}\int_0^{2\pi}\cos(nx)\cos(mx)dx &= 0 \quad \text{when} \quad m\neq n\\ \int_0^{2\pi}\cos^2(nx)dx &= \pi\\ \int_0^{2\pi}\sin(nx)\sin(mx)dx &= 0 \quad \text{when} \quad m\neq n\\ \int_0^{2\pi}\sin^2(nx)dx &= \pi\\ \int_0^{2\pi}\sin(nx)\cos(mx)dx &= 0 \quad \text{for any} \quad m,n\end{aligned}$$

These integrals are can be written more compactly using a symbol $\delta_{m,n}$ which is called the Kronecker delta.

$$\begin{aligned}\delta_{m,n} &= 1 \quad \text{when} \quad m=n\\ &= 0 \quad \text{otherwise}\end{aligned}$$

so then we can write

$$\begin{aligned}\int_0^{2\pi}\cos(nx)\cos(mx)dx &= \pi\delta_{m,n}\\ \int_0^{2\pi}\sin(nx)\sin(mx)dx &= \pi\delta_{m,n}\\ \int_0^{2\pi}\sin(nx)\cos(mx)dx &= 0\end{aligned}$$

Going back to our expression for $\cos(mx)f(x)$

$$\cos(mx)f(x) = \frac{a_0}{2}\cos(mx) + \sum_{n=1}^{\infty} a_n \cos(mx)\cos(nx) + b_n \cos(mx)\sin(nx)$$

we see that these integrals are telling just what we hoped for: every single term except for $a_m \cos(mx)\cos(mx)$ is going to integrate to zero over a full period:

$$\begin{aligned}\int_0^{2\pi} \cos(mx)f(x)dx &= 0 + \int_0^{2\pi} a_m \cos(mx)\cos(mx)dx \\ &= \pi a_m\end{aligned}$$

So we've found an expression that applies for any a_m constant! Rearranging, and swapping to using the n label (since we aren't using it for anything else any more), we get

$$\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx}$$

Let's compare this with our expression for a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

...we see that actually the a_n expression covers this too, since $\cos(0\ x) = 1$. This is why we put that extra factor of $1/2$ into the defintion of a_0, so that things would work out neatly and we wouldn't have to keep writing a separate line for a_0. BUT when we later come to work out a_n for a real function, we often need work out a_0 as a separate case, because for $n > 0$ we'll need to do things like dividing by n.

By exactly the same argument, we can get any/all of the b_n terms by multiplying $f(x)$ by the appropriate sine term and integrating.

So let's summarize the definition of the Fourier series and the expressions we've worked out for a_n and b_n. These are the crucial equations that we'll

often use

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$
$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

CRUCIAL:
Remember that we can use **any other complete period** for the integrals, e.g. $-\pi$ to π instead of 0 to 2π. We should use whatever is easiest.

1.5 A way to think of all this...

As an aside, we might think about a previous time in mathematics that we have met sometime similar: namely when we thought about vectors, and writing a vector in terms of components.

Comparison Table		
Feature	Vectors	Fourier Series

1.6 When our function has a period other than 2π

So far we've assumed that the periodic function we've been given has a period 2π. Suppose that we've now been given a function $g(x)$ that has some other period, L. So then $g(x) = g(x+L)$. How can we see how to scale the previous equations for the Fourier series of a function to apply here?

We'll let's define a new function $f(x)$ as follows:

$$f(x) \equiv g\left(\frac{L}{2\pi}x\right) \qquad \text{so that} \qquad g(x) = f\left(\frac{2\pi}{L}x\right).$$

What is the period of this new function?

The period of our new $f(x)$ is 2π, as we can confirm from

$$\begin{aligned} f(x+2\pi) &= g\left(\frac{L}{2\pi}x + \frac{L}{2\pi}2\pi\right) \\ &= g\left(\frac{L}{2\pi}x + L\right) = g\left(\frac{L}{2\pi}x\right) = f(x). \end{aligned}$$

But then the Fourier series for this new $f(x)$ must be given by the expressions we've already worked out:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx$$

However, we know how to get $g(x)$ from $f(x)$, it's just $g(x) = f\left(\frac{2\pi}{L}x\right)$. So

$$g(x) = f\left(\frac{2\pi}{L}x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)$$

and we can also translate the a_n expressions to use $g(x)$ instead of $f(x)$:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} g\left(\frac{Lx}{2\pi}\right)\cos(nx)dx \\ &= \frac{2}{L}\int_0^{L} g(u)\cos\left(\frac{2n\pi u}{L}\right)du \quad \text{changing variable } u \equiv \frac{Lx}{2\pi} \end{aligned}$$

And similarly for b_n. Let's collect this up. We'll write the variable of integration as x again because, well, we may as well! (Since it's just the label for the integral - it can be anything we like.) So:

If the period isn't 2π and instead $g(x) = g(x + L)$

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L}nx\right) + b_n \sin\left(\frac{2\pi}{L}nx\right)$$

$$a_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{2\pi}{L}nx\right)dx$$

$$b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{2\pi}{L}nx\right)dx$$

Note the integrals can use any complete period. Often $-L/2$ to $L/2$ is good.

1.7 Fourier Series for Even and Odd Functions

Suppose we are given some periodic function $g(x) = g(x + L)$ that has the special property that it is *odd*, i.e. $g(-x) = -g(x)$. This property will simplify things. To see why let's write out the coefficients using symmetric limits:

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} g(x)\cos(2n\pi x/L)dx$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} g(x)\sin(2n\pi x/L)dx$$

We notice immediately that a_0 is the integral of an odd function between symmetric limits, and so must be zero. What about the other a_n and the b_n?

For the other a_n, the integrand is the product of $g(x)$, which is an *odd* function, and a cosine which is always *even* function. The two functions multiplied together is itself an *odd* function, so all the a_n terms must be zero!

So if $g(x)$ is *odd* then we can simplify our Fourier series:

If the function $g(x)$ is odd...

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(2n\pi x/L)$$

Also notice that the integrand of b_n is even, so we can just do half the integral, from 0 to $L/2$, and double it.

What about if we are given an $g(x)$ that is an even function, $g(x) = g(-x)$? Then the integrand of the expression for b_n is an *odd* functions, because it is a product of $g(x)$ (even) and a sine function (odd). So all the b_n must be zero. If $g(x)$ is *even* then we can simplify our series to

If the function $g(x)$ is even...

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2n\pi x/L)$$

Again, notice that the integrand of a_n is now going to be even, so we can just do half the integral, from 0 to $L/2$, and double it.

1.8 An Example: The Square Wave

Consider the periodic function $f(x) = f(x + 2\pi)$ where

$$f(x) = +1 \quad \text{for} \quad 0 < x < \pi \qquad \text{and} \qquad f(x) = -1 \quad \text{for} \quad \pi < x < 2\pi.$$

Sketch the function here

The period is 2π so we can use the simple expressions, rather than the "L" ones. More importantly, $f(x)$ is an *odd* function, so things are simplified!

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin(nx) \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \end{aligned}$$

So let's work out the b_n. The following space is left blank for you to fill the derivation:

So we find that

$$b_n = \frac{2}{\pi}\left(\frac{1-(-1)^n}{n}\right)$$

but this just alternates between zero for even n, and $4/(n\pi)$ when n is odd. So there are two equivalent ways we can write the series:

$$f(x) = \frac{4}{\pi}\sum_{n=1,\ n\ odd}^{\infty}\frac{\sin(nx)}{n} \;=\; \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{2n+1}$$

One method explicitly states that the sum runs over only odd n, the other runs over all n but uses $2n+1$ to make odd numbers. They're equally good.

Computer exercise:
Use your maths software to see if our expression for the Fourier series of the square wave works! But we'll have to cut off the series at some finite maximum N so we can see what it looks like to cutoff at different points. If you're using matlab, here's a basic script for $N = 10$:

```
f=0;
N=9;
syms x
for n=1:2:N
    a = 4/(pi*n);
    f = f + a*sin(n*x);
end

fplot(f,[-5,5])
```

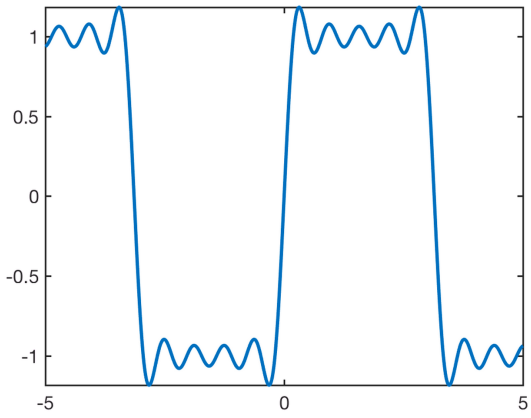

Figure 1.5: Example of what matlab should produce for $N = 10$.

Computer extra exercise:
Expore: As we increase the number of terms in the series, what happens to those "overshoots" that occur where the corners of the square wave should be?
(Actually that overshoot even has a name: 'the Gibbs Phenomenon'.)

1.9 Second Example: The Triangular Wave

Consider the periodic function $g(x) = g(x + 2\pi)$ where

$$g(x) = \pi - |x| \qquad -\pi \leq x < \pi$$

Sketch the function here

Again the period is 2π so we can use the simple expressions, rather than the "L" ones. More importantly, $g(x)$ is an *even* function, so things are again simplified, this time by $b_n = 0$. So:

$$\begin{aligned} g(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx) dx \end{aligned}$$

So let's work out the a_n terms. As usual, we need to take the a_0 case separately from the $n > 0$ cases, because of stuff like dividing by n.

We will have to deal with $|x|$ which is a tricky function. But there is an easy solution: write $\pi - |x|$ as just $\pi - x$ when $x > 0$, and $\pi + x$ when $x < 0$. And in fact, since the integrand of a_n is *even*, we can just do the positive half of the integral and double it!

$$a_0 = \frac{2}{\pi} \int_0^{\pi} g(x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right) = \pi$$

Now we should ask ourselves, as a check, 'does this look right?' This constant offsets the whole graph upward. Seems about right.
Time to look at the general a_n, for $n > 0$:

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(x)\cos(nx)dx \quad \text{because we are doubling the positive half}$$

Finally we obtain

$$a_n = \frac{2}{\pi}\left(\frac{1-(-1)^n}{n^2}\right)$$

So the first few terms of the series are

$$\frac{4}{\pi}\frac{\cos(x)}{1}, \quad 0, \quad \frac{4}{\pi}\frac{\cos(3x)}{9}, \quad 0, \quad \frac{4}{\pi}\frac{\cos(5x)}{25}, ...$$

Now we are ready to write out the answer:

$$g(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}$$

where here we've opted for the '$2n+1$' trick to create just the odd numbers.

Computer exercise:
Verify that indeed our expression is correct, so that as we add more and more terms we get closer to the perfect triangular wave. You can reuse the code that we employed for the square wave, but modify it to the present case.