

Fourier Series, Fourier Transforms, and PDEs

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Week 2: Complex Fourier Series & Transforms

In this second week of the course we'll look through some further features of Fourier series. Then we'll step up to Fourier transforms, which are even more powerful and useful.

1.1 Integration and differentiation of Fourier Series

Look again at the two examples we've just examined, the square wave (call it $f(x)$) and the triangular wave (call it $g(x)$):

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1}$$

$$g(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}$$

We can see from inspection of the graphs that these two functions are actually related: the square wave is (minus) the derivative of the triangular wave.

So presumably, it must be the case that if we differentiate Fourier series for $g(x)$ term by term, will we get $f(x)$? Let's try.

$$\begin{aligned} \frac{d}{dx} g(x) &= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{d}{dx} \frac{\cos((2n+1)x)}{(2n+1)^2} \\ &= \frac{-4}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1} \end{aligned}$$

And this is exactly (minus) the expression we found for the square wave. In general, for the types of function you will meet, differentiation and integration of Fourier series 'works'. One thing to watch out for is the case of differentiating a piecewise function that has discontinuous jumps - what happens there can be more complex.

Matlab exercise:

Use matlab to try differentiating the square wave and plot that: look at the points where the original function $f(x)$ has its jumps.

```
f=0;
N=10;
syms x
syms pi
for n=1:2:N
    a=4/(pi*n);
    f = f + a*sin(n*x);
end
fplot(f, [-5, 5])

h=diff(f, x)
fplot(h, [-5, 5])
```

1.2 Functions Only Defined in a Finite Range

In many real-world situations, we might want to deal with a function that is only defined in a limited range. It's easy to think of an example. Imagine a function that represents the vertical displacement of a guitar string away from its rest position. Suppose the string-at-rest lies along the x -axis from $x = 0$ to $x = L$. We'll consider the situation where the guitarist has plucked the string in the middle and the string is just about to be released so that, for a moment, it has a triangular form:

$$\begin{aligned} D(x) &= \frac{2d}{L} |X| & \text{for } 0 \leq x \leq \frac{L}{2} \\ &= \frac{2d}{L} (L - |X|) & \text{for } \frac{L}{2} < x \leq L \end{aligned}$$

The point is that the function is simply *not defined* for $x < 0$ or $x > L$ since this is beyond the extent of the guitar!

What should we do here? Well we are free to specify the function *any way we like* outside of the given range! **So: can can just extend the definition in such a way that the function becomes periodic.** Of course the smart choice will be to specify it in a way that makes the Fourier

series as simple as possible. Maybe we'll see a way to make the extended function even (then the Fourier series will only need the constant and the cosine terms) or odd (only sine terms needed).

Warning: In any given real modelling problem we will need to think a bit further about the proper way to extend our function and make it periodic. We'll see more about this in the next part of the course, but for now here's a rule of thumb: Don't put a discontinuity (a jump) in function at the point where it goes out of the range of the original definition. Keep it continuous, e.g. by mirroring the defined part of the function at its end points.

For the above example, let's sketch how the plucked guitar string can be part of a suitable infinite periodic function:

1.3 Fourier Theorem: Dirichlet Conditions

Earlier when we obtained expressions for the coefficients a_0 , a_m and b_m , by starting from the **assumption** that the periodic function $f(x)$ could be expressed perfectly as an infinite series of $\cos()$ and $\sin()$ terms. Is this assumption always true?

The answer is that it is true in all cases that you will meet! But there are extreme cases that do not work. Formally, we can write out the Dirichlet Conditions which state when a function $f(x)$ will have an exact Fourier series.

The Dirichlet Conditions

If a single-valued function $f(x)$ is periodic with period 2π , and in one period it

- has a finite number of maxima and minima,
- has a finite number of discontinuities, and
- its absolute value is integrable,

then our usual expression for the Fourier series applies. In other words,

the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ will indeed match $f(x)$ when

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx.$$

Where $f(x)$ has a discontinuity then the Fourier series converges to the midpoint of the jump.

Matlab exercise: Try to think of a function that would NOT meet the Dirichlet Conditions. Use matlab to check if you're right.

1.4 Parseval's Theorem and Errors in the Fourier Series

Parseval's theorem, which dates back to a 1799 and the work of Marc-Antoine Parseval, can be thought of as a way to relate the square of a function to its Fourier series. At first sight it may not look as if there is anything of very great practical significance to it. We can state it like this:

$$\frac{1}{2\pi} \int_0^{2\pi} (f(x))^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

Note that the left hand side of the equation is really taking an average of the function since it is integrating it over a full period and dividing by the

period length. So in words, we can say that the average value of a function *squared* over one period can be written as just a certain sum of the squares of the function's Fourier coefficients (note the $\frac{1}{2}$ in front of the $n \neq 0$ terms).

We could prove this pretty easily if we wanted - let's outline how. The proof would go like this. Consider the expression for $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

and think about the kind of terms that we'd see if we went to the trouble of squaring this expression. Well there would be a lot of actual squared terms, including everything of the form $(a_m \cos(mx))^2$ and $(b_m \sin(mx))^2$. Integrating all these terms over one period and dividing by 2π would give us exactly the expression we want. Meanwhile, all the *cross terms* in $f(x)$ squared, like $2 \cos(mx) \cos(nx)$ for $n \neq m$ will integrate to zero. (You showed that integrals like that are zero, on Problems Sheet 1).

But why might Parseval's theorem be interesting to us? It helps us by saying something about errors in a Fourier series. Suppose that someone gave us a Fourier series, supposedly the series for some function $f(x)$, but there are *errors*, so really the series makes some function $E(x)$. Let's write the true coefficients as a_n, b_n , and the incorrect ones as A_n, B_n .

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \text{ but} \\ E(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx) \end{aligned}$$

Now, how badly wrong is the incorrect function? One way to measure

how wrong it is, is just to work out the total integral of the square of the difference, over one period. But Parseval's Theorem makes that easy:

$$\int_0^{2\pi} (E(x) - f(x))^2 dx = 2\pi \left(\frac{(a_0 - A_0)^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - A_n)^2 + \sum_{n=1}^{\infty} (b_n - B_n)^2 \right)$$

So the total error is made up of the squares of all the individual errors. Because the errors are squared, they are positive - and so one error can't help to cancel another. Here are a couple of consequences:

- We already knew that if one of our Fourier coefficients is wrong, it will result in the overall Fourier series failing to match the target function. But now we can say it doesn't even help if we try to adjust another Fourier coefficient to compensate (it would just make it worse).
- This also means that if we know we are going to cut off our Fourier series after (say) ten terms in some application, still there is no point in trying to 'boost' the first term terms to compensate for the later missing ones.

1.5 Complex Form of Fourier Series

There is another way to write a Fourier series - a way that is actually more elegant from a mathematical point of view.

For a function $f(x)$ having period 2π , so $f(x) = f(x + 2\pi)$, we can write:

$$\begin{aligned} f(x) &= c_0 + c_1 \exp(ix) + c_{-1} \exp(-ix) + c_2 \exp(2ix) + c_{-2} \exp(-2ix) + \dots \\ &= \sum_{n=-\infty}^{\infty} c_n \exp(inx) \end{aligned}$$

Where the coefficients, including the c_0 , are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \exp(-inx) dx.$$

An exercise on Problem Sheet 2 shows how to derive this expression for c_n . Here let's quickly derive the relation between the Fourier series we are used to, and this new form. Well of course we'd usually write:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

We can use the identities

$$\cos(u) \equiv \frac{1}{2} (\exp(iu) + \exp(-iu)) \quad \sin(u) \equiv \frac{1}{2i} (\exp(iu) - \exp(-iu))$$

so see that

$$c_n = \frac{a_n}{2} + \frac{b_n}{2i} = \frac{1}{2}(a_n - ib_n) \quad \text{for } n > 0, \quad \text{and} \quad c_n = \frac{1}{2}(a_n + ib_n) \quad \text{for } n < 0$$

This leaves just the constants in the two expressions, $c_0 = a_0/2$.

Now just for completeness, let's write out the expressions for a complex Fourier series when the period is L ,

For a function $f(x)$ having period L , so $f(x) = f(x + L)$, we can write:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(i \frac{2n\pi x}{L}\right)$$

where

$$c_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} g(x) \exp\left(-i \frac{2n\pi x}{L}\right) dx.$$

(Or we can integrate over any other complete period of course).

These expressions can be obtained from our previous expressions for a 2π period function, just by scaling as we did before for regular Fourier series.

1.6 Introducing the Fourier Transform

Before moving on to talk about PDE's, we should take the step beyond Fourier *Series* to something that's even more powerful and important: the Fourier *Transform*. There are many ways to motivate this topic, which is a kind of upgrade, but here we will just summarise the core idea.

Let's start by rewriting the expressions for the complex Fourier series with a small change of notation

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{L} F(k_n) \exp(i k_n x) \quad \text{where} \quad k_n \equiv \frac{2\pi n}{L}.$$

Here we've replaced the constants c_n using a function F that *generates* the constants – for any value of k we define the new function F as

$$F(k) = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \exp(-ikx) dx.$$

Note that k_n is simply the *spatial frequency*, i.e. the number of cycles per metre multiplied by 2π .

The difference between two consecutive allowed k values is $k_{n+1} - k_n$ which is $\frac{2\pi}{L}$, and we'll represent this with the symbol $\delta k \equiv \frac{2\pi}{L}$.

Summarising, in this new notation we can write:

Using Fourier **Series**, a periodic function $f(x) = f(x + L)$ is

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(k_n) \exp(i k_n x) \delta k$$

$$\text{with } F(k_n) = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \exp(-i k_n x) dx$$

$$\text{using } k_n \equiv \frac{2\pi n}{L} \text{ and } \delta k \equiv \frac{2\pi}{L}$$

This definition of a Fourier series is very compact, with a lot of similarity between the expressions for $f(x)$ and $F(k_n)$. But it is just a change of notation, we haven't added anything new.

Now let's think about what happens if we let the period of the function become longer and longer, i.e. $L \rightarrow \infty$. Well, of course the limits of the $F()$ integral will become $\pm\infty$. The more interesting thing is to look at the sum expression. As L tends to infinity, the quantity δk becomes infinitesimal – essentially, the k_n quantity becomes a continuous variable k and we arrive at the Fourier transform. Crucially, now $f(x)$ **doesn't need to be periodic**.

Using the Fourier **Transform** $F(k)$ we can write *any* function $f(x)$ as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(i k x) dk \text{ where } F(k) = \int_{-\infty}^{\infty} f(x) \exp(-i k x) dx$$

It may be helpful to pause and compare the two blue boxes above to see how we've turned the series into the integral.

Note that there are various slightly different definitions that move around the 2π , one common example is :

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \exp(2\pi i x \xi) d\xi \quad \text{where} \quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi i \xi x) dx$$

The important thing to appreciate is that the function $F(k)$ does a similar job to the constants c_n in the complex Fourier series - it tells us how much of each frequency makes up $f(x)$.

Aside: The function $F(k)$ can be complex, giving both real and imaginary parts, even if $f(x)$ is strictly real. The magnitude $|F(k)|$ tells us *how much* of a given frequency is present in $f(x)$, but what meaning could the phase have?

Notice how similar the two expressions are; if we are given *either* $f(x)$ or $F(k)$ then we can obtain the other one. Really they are telling us the same thing in two different ways: $f(x)$ tells us how some quantity varies in space, while $F(k)$ tells us how the same function varies in frequency.

Let's do an example. Suppose we have a function that is non-periodic and is defined like this:

$$\begin{aligned} f(x) &= 1 - 2|x| \quad \text{for } |x| \leq \frac{1}{2} \\ &= 0 \quad \text{for } |x| > \frac{1}{2}. \end{aligned}$$

Let's sketch this and then find the Fourier transform $F(k)$:

Finally we obtain,

$$F(k) = 4 \frac{1 - \cos(k/2)}{k^2}.$$

Given this $F(k)$ we can use matlab or similar to check that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(i k x) dk$$

does indeed give us the function we want! The code is shown in the following blue box.

In the code below, notice we integrate between limits $k = \pm 10$ instead of $\pm\infty$ because it is a numerical integral and $k = \pm 10$ will be plenty. But what will happen if we reduce the limits to, say, $k = \pm 1$?

Matlab code:

```
syms k
syms pi

F=4*(1-cos(k))/k^2

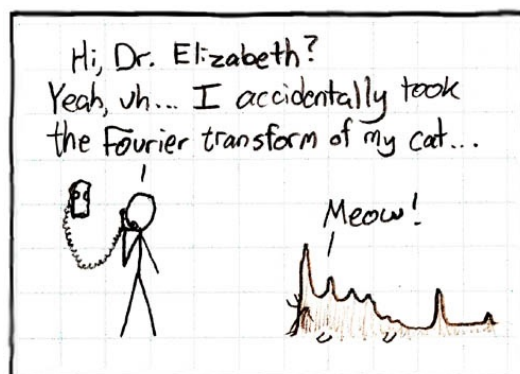
% Make a list of x values we want to plot at
xVals=linspace(-3,3,31)

% Make a list of the corresponding integrands
% note matlab uses j for sqrt of -1
fns=F*exp(j*xVals*k);

% Do all the corresponding integrals!
% This may take a minute to perform.
output=vpa(int(fns,k,-10,10));

% Check imaginary part is negligible, remove it
max(abs(imag(output)))
output=real(output);

plot(xVals,output,'-o')
```



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