Fourier Series, Fourier Transforms, and PDEs

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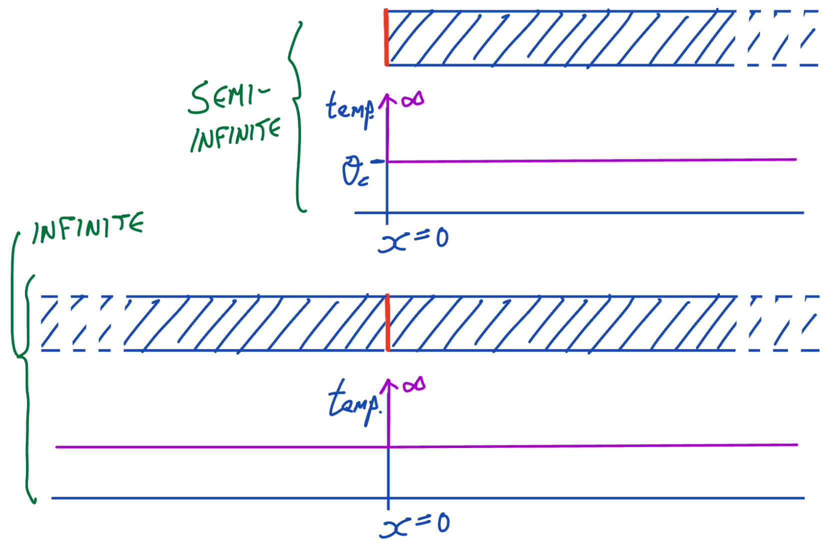
Week 4: More diffusion, and then Waves

1.1 Introduction

In this final set of notes, we will take a look at advanced (challenging!) problems in diffusion before moving on to think about a more straightforward treatment of waves.

1.2 An instantaneous pulse of energy

Suppose that we have a bar of metal which had been at a uniform temperature Θ_c, but at $t = 0$ it receives a pulse of energy at one end – as if the end surface were subjected to a short, powerful laser pulse. Let’s make the simplifying assumption that at the $t = 0$ instant, the new heat energy is all exactly at $x = 0$. Then in that initial moment the very concept of temperature is not well defined (it will be infinite) – but we can still expect that only a certain finite amount of heat *energy* has entered the material; call it ΔE.



Before we get going, let’s first make the problem more symmetric: instead of all the heat being at the $x = 0$ end of a “semi-infinite” bar that exists for zero and all positive x, let’s consider the related problem of a fully infinite bar, running from $-\infty$ to ∞, and at $t = 0$ we'll assume there is a spike of heat energy exactly at $x = 0$. In fact the future heat distribution for $x \geq 0$ is going to be the *same*, up to a factor of 2, in both cases. Why?

To model the temperature mathematically, we want an infinitely narrow, infinitely tall spike which, however, has a certain finite area under it. Fortunately the thing we need exists and is well-studied, it is the Dirac delta function:

$$\begin{aligned} \delta(x) &= \infty \text{ if } \quad x = 0 \\ \delta(x) &= 0 \quad \text{if } \quad x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\, dx &= 1 \end{aligned}$$

Using this we can write our $t = 0$ limit as

$$\Theta(x, t = 0) = \Theta_c + \frac{\Delta E}{C_L}\,\delta(x) \tag{1.1}$$

Here C_L is the heat capacity of our bar per unit length. Using the defining properties of the Direct delta function, we see that indeed a certain finite amount of (extra) energy ΔE is present exactly at the $x = 0$ location.

That's a neat way to write the initial condition, but to make progress we need to express it differently. We've previously been using Fourier Series methods, but this time the initial problem isn't periodic, and it isn't defined over a finite range, so we can't expect to use a Fourier Series method to re-express the initial condition. But we have the tool we need for such a situation: The more powerful Fourier *Transform*.

Let's express the initial condition using a Fourier Transform. We're already comfortable with the Θ_c constant (we can guess that will just carry right through to the time-dependent solution), but we do want to re-express that $\delta(t)$ Dirac delta function. Recall the definition of a Fourier transform:

Using the Fourier **Transform** $F(k)$ we can write *any* function $f(x)$ as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp(i\,k\,x)dk \quad \text{where} \quad F(k) = \int_{-\infty}^{\infty} f(x)\exp(-i\,k\,x)dx$$

So in the present case, for $f(x) = \delta(x)$ we need to find $F(k)$. We write

$$F(k) = \int_{-\infty}^{\infty} \delta(x) \exp(-i\,k\,x) dx \tag{1.2}$$

which looks challenging but in fact it is super easy: it turns out that for any function, call it $g(x)$, when we multiply by the Dirac delta function and integrate over any range that includes the origin $x = 0$, the integral is just $g(0)$. That's so handy and powerful that we should write it out:

For any function $g(x)$ that is well-behaved at $x = 0$, and any limit $a \neq 0$,

$$\int_{-a}^{a} \delta(x) g(x) dx = g(0)$$

Thus we can immediately follow on from Eqn.(1.2), by noting that this is just a case with $g(x) = \exp(-ikx)$, and we get

$$F(k) = \exp(-i\,k\,0) = 1 \tag{1.3}$$

...so the Fourier transform of the Dirac delta function is just one ('unity'). Thus we can write

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\,k\,x) dk$$

and with this new way of writing the Dirac delta function, we can rewrite the $t = 0$ boundary condition of our problem, Eqn.(1.1) as

$$\Theta(x, t = 0) = \Theta_c + \frac{\Delta E}{2\pi C_L} \int_{-\infty}^{\infty} \exp(i\,k\,x) dk \tag{1.4}$$

OK so we expressed our Dirac delta function as an integral over $\exp(ikx)$ but how does that help? Recall that we know solutions to the diffusion equation include anything of the form

$$(A\cos(\xi x) + B\sin(\xi x)) \exp(-\alpha \xi^2 t)$$

for any values of A, B and ξ. But we've just found that our $t = 0$ case is an integral over an integrand which matches up with our allowed family of solutions if $A = 1$, $B = i$ and $\xi = k$. Can we just multiply in the corresponding $\exp(-\alpha k^2 t)$ term, *inside the integral*, and thus get a complete solution?

Yes! This gives the solution:

$$\Theta(x, t = 0) = \Theta_c + \frac{\Delta E}{C_L} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\alpha k^2 t) \exp(i\, k\, x) dk \tag{1.5}$$

This is in fact a perfectly fine solution to the problem, but in this case we can go further and perform the integral. Let's see how:

Finally, we conclude

$$\int_{-\infty}^{\infty} \exp(-\alpha k^2 t) \exp(i\, k\, x) dk = \exp\left(-\frac{x^2}{4\alpha t}\right) \sqrt{\frac{\pi}{\alpha t}}$$

and we can rewrite our solution from Eqn.(1.5) as

$$\Theta(x, t) = \Theta_c + \frac{\Delta E}{2C_L} \frac{1}{\sqrt{\pi \alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right)$$

For all $t > 0$, this is a Gaussian that gradually gets wider and less tall. Thus we have solved the "pulse of energy at point $x = 0$" problem. But by the way, we have also proved that any Gaussian initial distribution (which would just correspond to some particular $t = t_0$ value here) will simply spread and be less peaked over time – and that's the "guess" that we made in the last lecture; we've now derived the answer with no guessing required.

1.3 Heat or matter diffusion into an infinite region

It is an interesting and challenging problem to consider taking a massive object, initially all at a constant temperature, and raising (or lowering) the temperature of one surface. The idea here is that the object is so vast that, during the time we are interested in, it is effectively infinite. In abstract terms we consider a half-space: a region that has a boundary (like the $x = 0$ plane, say) but which extends infinitely far in the other directions. Mathematically, an identical problem arises in the caburization of steel. A surface layer of steel is hardened by carbon which diffuses into the steel from a carbon rich atmosphere: again we assume the steel is deep enough that carbon xwill never diffuse to the far side. Sketch this situation:

Our diffusion equation applies, of course, either

$$\frac{\partial \Theta}{\partial t} = D\frac{\partial^2 \Theta}{\partial x^2} \quad \text{for heat diffusion, or} \qquad \frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \quad \text{for matter diffusion.}$$

1.4 Carburising of steel

Imagine an infinitely large block of material (steel) that exists in the space $x \geq 0$ - so the plane $x = 0$ is its surface. Suppose that initially the material contains some concentration c_{init} of a particular impurity (carbon). However, the surface of the material is always in contact with a high concentration of that element (maybe, a carbon-rich gas), so that *at the surface* of the material, i.e. at $x = 0$, the concentration is always at the high value of c_s. We can summarise the conditions like this:

Initial condition: $c(x, 0) = c_{\text{init}}$ for $x > 0$.

Boundary condition at $x = 0$: $c(x = 0, t) = c_s$.

Boundary condition as $x \to \infty$: $c(x, t) \to c_{\text{init}}$.

Now we will find it convenient to rewrite $c(x, t)$ like this:

$$c(x, t) = c_s + (c_{\text{init}} - c_s) S(x, t).$$

We know the form of $S(x, t)$ when $t = 0$:

$$S(x, t = 0) \equiv R(x) \text{ where}$$

$$\begin{aligned} R(x) &= -1 \quad x < 0 \\ R(x) &= 0 \quad\;\;\, x = 0 \\ R(x) &= +1 \quad x > 0. \end{aligned}$$

Notice that we've extended $c(x, t)$ into the $x < 0$ region; we're free to do this since it was undefined there before. Let's sketch $c(x, t = 0)$, and also how we expect the $c(x, t)$ to vary with x at a much later time. Also sketch $R(x)$.

We might notice that when we write $c(x,t) = c_s + (c_{\text{init}} - c_s)S(x,t)$ then the constant c_s and the function $(c_{\text{init}} - c_s)S(x,t)$ play the roles that we previously called the 'steady state' and the 'transient', although because we have an infinite block in the $+x$ direction, then for any finite t there will always be an x so large that $S(x,t) \approx 1$, i.e. it doesn't really vanish as the transients we've considered before.

If we substitute $c(x,t) = c_s + (c_{\text{init}} - c_s)S(x,t)$ into our diffusion equation, we find that constant c_s disappears and factor $(c_{\text{init}} - c_s)$ cancels, so

$$\frac{\partial S(x,t)}{\partial t} = D\frac{\partial^2 S(x,t)}{\partial x^2}.$$

Thus our challenge is to find an $S(x,t)$ that satisfies this equation and also the boundary condition $S(x,t=0) = R(x)$.

Now $R(x)$ cannot be written as a Fourier series, because it is not periodic and nor is it defined only in a finite range. However we know that we can describe such functions using the Fourier transform. Let's use the 'alternative' definition which is a bit neater for us here:

$$R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp(ikx)dk \qquad \text{where} \qquad F(k) = \int_{-\infty}^{\infty} R(x)\exp(-ikx)dx.$$

(We could also use the definition based on ξ if we wanted – it ends up the same of course). Then:

$$F(k) = \int_{-\infty}^{0}(-1)\exp(-ikx)dx + \int_{0}^{\infty}(+1)\exp(-ikx)dx$$

and these integrals are a bit tricky. We can use what's called an 'integrating factor':

$$\begin{aligned}
F(k) &= \lim_{\alpha\to 0}\left\{\int_{-\infty}^{0}(-1)\exp\left((\alpha - ik)x\right)dx + \int_{0}^{\infty}(+1)\exp\left((-\alpha - ik)x\right)dx\right\} \\
&= \lim_{\alpha\to 0}\left\{-\left[\frac{\exp\left((\alpha - ik)x\right)}{\alpha - ik}\right]_{-\infty}^{0} + \left[\frac{\exp\left(-(\alpha + ik)x\right)}{-\alpha - ik}\right]_{0}^{\infty}\right\} \\
&= \lim_{\alpha\to 0}\left\{\frac{-1}{\alpha - ik} + \frac{1}{\alpha + ik}\right\} = \frac{-2i}{k}.
\end{aligned}$$

A simple answer once we get there! But then

$$\begin{aligned}
R(x) &= \frac{-i}{\pi}\int_{-\infty}^{\infty}\frac{\exp(ikx)}{k}dk = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{-i\cos(kx) + \sin(kx)}{k}dk \\
&= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(kx)}{k}dk \qquad (1.6)
\end{aligned}$$

where we have used the fact that $\cos(xk)\frac{1}{k}$ is (even) times (odd) and so must be zero when integrated between symmetric k limits, and $\sin(xk)\frac{1}{k}$ is (odd) times (odd) and thus even, so its integral is just twice the positive part.

So: we've got a rather elegant and compact expression for the function $R(x)$, which was just the $t = 0$ limit of $S(x,t)$. Last time when we managed to get the $t = 0$ boundary condition into Fourier transform format, we just inserted the same time-dependent factor as *would have* worked in the Fourier series case, right into our integrand. We can just do so again, multiplying in $\exp(-Dk^2t)$.

$$S(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(kx)}{k}\exp(-Dk^2t)dk \tag{1.7}$$

This is a perfectly good solution, as we could confirm by putting it into the diffusion equation (the partial derivates would differentiate inside the integral sign, which is fine). We can therefore write down the complete solution as $c(x,t) = c_s + (c_{\text{init}} - c_s)S(x,t)$. **We could just stop here!** Let's pause and use matlab to see how it looks.

Matlab check: Investigate our solution $c(x,t)$.

However we can neaten up our solution to make it more compact:

$$\begin{aligned} S(x,t) &= \frac{2}{\pi}\int_0^\infty \frac{\sin(\eta\frac{x}{\sqrt{Dt}})}{\eta}\exp(-\eta^2)d\eta \quad \text{using} \quad \eta \equiv k\sqrt{Dt} \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin(\beta\eta)}{\eta}\exp(-\eta^2)d\eta \quad \text{using} \quad \beta \equiv \frac{x}{\sqrt{Dt}}. \end{aligned}$$

This form is elegant since both x and t have been subsumed into the variable β which itself appears only at one point in the expression.

Can we get any further and actually perform the integral? Not exactly, but it can be transformed into an even more simple looking integral – in the appendix we show that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(\beta\eta)}{\eta}\exp(-\eta^2)d\eta = \frac{2}{\sqrt{\pi}}\int_0^{\beta/2}\exp(-u^2)du. \tag{1.8}$$

Notice that the physical variables x and t now appear only in the upper limit of the integral. Now this integral on the right has a name: the *error function*.

$$\text{erf}(q) \equiv \frac{2}{\sqrt{\pi}}\int_0^q \exp(-u^2)du$$

So in fact we have found that

$$S(x,t) = \mathrm{erf}(\beta/2) = \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

Remembering that $c(x,t) = c_s + (c_{\text{init}} - c_s)S(x,t)$, we can write our full solution as

$$c(x,t) = c_s + (c_{\text{init}} - c_s)\,\mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right).$$

Now there's also something called the 'complimentary error function' defined simply as $\mathrm{erfc}(q) \equiv 1 - \mathrm{erf}(q)$. We could rearrange to write:

$$c(x,t) = c_{\text{init}} + (c_s - c_{\text{init}})\,\mathrm{erfc}\left(\frac{x}{2\sqrt{Dt}}\right).$$

This is a nice way around to write it, since it emphasises how the concentration is going up from c_{init} to its actual value at some x and t. But remember its just an alternative way of writing the answer that we knew instantly after we'd found the Fourier transform of the initial distribution, i.e. Eqn. (1.7). Since then all we've done is rework it into other forms.

Matlab exercise:
Use matlab to investigate this solution and confirm it. In matlab **erfc( )** is the complimentary error function. Try $\phi_1 = 1$, $c_{\text{init}} = 0$, $c_s = 1$, $D = 1$ and $t = 1$, for example.

```
cInit=0;
cS=1;
D=1;
t=1;
syms x
f=cInit+(cS-cInit)*erfc(x/(2*sqrt(D*t)))
myPlot=fplot(f,[-10,10]);
```

Before moving on its worth thinking about the fact that our solutions reveal that the ratio $x/\sqrt{Dt}$ is the crucial thing, rather than x and t separately. Roughly speaking this means, if you want the carbon to penetrate twice as far into the steel you need to wait four times as long!

To take a specific example: at a temperature of $1,000\,\mathrm{K}$ the diffusion coefficient of steel is about $2 \times 10^{-12}\,\mathrm{m}^2/\mathrm{s}$. How long will it take to reach carbon concentration of $c = 0.01$ at a depth of $100\,\mu$m when the material is initially carbon-free and the external gas is 10% carbon?

$$c_{\text{init}} = 0 \quad c_s = 0.1 \quad x = 10^{-4} \quad D = 2 \times 10^{-12}.$$

$$c(x,t) = 0.1\,\text{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)$$

But we want $c(x,t) = 0.01$

$$\text{erfc}\left(\frac{x}{2\sqrt{Dt}}\right) = \frac{0.01}{0.1} = 0.1$$

Use a scientific calculator, or matlab, or a table of numbers (online) to find inverse-erfc:

$$\frac{x}{2\sqrt{Dt}} = 1.163$$

$$t = \frac{1}{4D}\left(\frac{x}{1.163}\right)^2 = \frac{1}{8 \times 10^{-12}}\left(8.60 \times 10^{-5}\right)^2 = 925\,\text{sec}$$

Here's a summary table showing the various solutions we've looked at for **Fick's 2nd Law** and **Heat flow**. You may want to sketch in some graphs.

Initial condition	**Physical example**	**Analysis method**	**Solution for general time**
Infinite square wave or similar	Stack of layers/films intermix	Fourier series for $t = 0$ then insert $\exp(-k^2 t)$ for each term	Higher Fourier terms decay faster; creates smoothing
Any case with fixed end-points (all t)	Sheet/bar between 2 regions held @ diff. temps or concentrations	Make a periodic function (carefully!) and use Fourier Series as above	As above
Gaussian in middle or at boundary	Cooling after localised heating	Guess: 'it stays Gaussian'; Solve by substitution	Gaussian width increases as $\sqrt{t}$
Infinitely dense (or hot) layer in middle or at boundary	Heat pulse; or film of diffusive material	Fourier Transform of Dirac delta function; then apply $\exp(-k^2 t)$ inside integral	Becomes a Gaussian as in above case
Infinite block abuts reservoir; or two infinite blocks abut	Carburisation of steel	Fourier Transform of step function; then apply $\exp(-k^2 t)$ inside integral and simplify (hard!)	Solution involves the Error function; Diffuses $\sim \sqrt{t}$

New Topic: The Wave Equation

1.5 Derivation of the Equation for a Plucked String

Consider an elastic string, fixed to a board at two points, distance L apart. The density of the string (mass per unit length) is ρ, and the tension in the string is T. Let's sketch it:

We assume that the amplitude of the transverse motion of the string is small, and therefore that all motion of the string is in the y-direction. We consider two points, P and Q, at x and $x + \delta x$. Resolving forces on the segment PQ,

$$T_1 \cos(\alpha) = T_2 \cos(\beta) = T$$

in order that there should be zero net force in the x-direction and therefore no x-motion is introduced. But then

$$T_1 = T \sec(\alpha) \qquad\qquad T_2 = T \sec(\beta)$$

Now, resolving forces on segment PQ vertically:

$$T_2 \sin(\beta) - T_1 \sin(\alpha) = \rho\ \delta x \frac{\partial^2 y}{\partial t^2}$$

This is just *force* = *mass* times *acceleration*. But using our expressions for T_1 and T_2, we can then write

$$T \tan(\beta) - T \tan(\alpha) = \rho\ \delta x \frac{\partial^2 y}{\partial t^2} \tag{1.9}$$

So the trick will be to evaluate this difference of tan() functions. What is tan()? It is simply the ratio of the vertical increment to the horizontal increment. So, since the angle the curve $y(x)$ makes with the x-axis is α at point x, then

$$\tan(\alpha) = \frac{\partial y}{\partial x}$$

at that point x. Let's write

$$g(x) \equiv \frac{\partial y}{\partial x}$$

where we've chosen g for 'gradient'. Then $\tan(\alpha) = g(x)$ and meanwhile, further along at point $x + \delta x$ we write

$$\begin{aligned} \tan(\beta) &= g(x + \delta x) \\ &= g(x) + \frac{\partial g}{\partial x}\delta x \\ &= g(x) + \frac{\partial^2 y}{\partial x^2}\delta x \end{aligned}$$

But then

$$T(\tan(\beta) - \tan(\alpha)) = T\frac{\partial^2 y}{\partial x^2}\delta x = \rho\ \delta x\frac{\partial^2 y}{\partial t^2} \tag{1.10}$$

So equating Eqn. (1.10) with Eqn. (1.9) and cancelling terms we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}.$$

and this is our wave equation. But we might like to replace the ratio of two constants by a single constant. Looking at the dimensions of the two terms, we see that the constant T/ρ must have the dimensions of (space squared) over (time squared), i.e. it is a speed squared. Writing $c \equiv \sqrt{T/\rho}$ as this wave speed, we have

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}.$$

1.6 D'Alembert's Formula

Before moving on to find solutions to the wave equation for interesting cases, we'll pause to look at the wave equation and consider what happens when we make a change of variables: Let $u = x + ct$ and $v = x - ct$. This approach is often associated with Jean-Baptiste le Rond d'Alembert, 1717-1783.

$$\frac{\partial y(x,t)}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

Since we didn't make any assumptions about y to obtain this, we can write

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}.$$

Similarly

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right).$$

And again we can write

$$\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right).$$

Then

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

and similarly

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

But our wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \Rightarrow \qquad \frac{\partial^2 y}{\partial u \partial v} = 0$$

and therefore

$$y = f(u) + g(v) = f(x + ct) + g(x - ct)$$

What is this telling us?

1.7 A plucked string: The general solution

Consider a string of length L, with mass per unit length ρ and tension τ. We assume that we are given some initial conditions, and we want to calculate the subsequent profile of the string $y(x, t)$. We know

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad\qquad c^2 \equiv \frac{\tau}{\rho}.$$

As usual we start by looking at the case of a separable solution: In other words, we assume $y_{\text{sep}}(x, t) = X(x)T(t)$. Then

$$X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2} \qquad\qquad \Rightarrow X\frac{d^2 T}{dt^2} = c^2 T\frac{d^2 X}{dx^2}$$

Dividing through by XT we have

$$\frac{1}{T}\frac{d^2T}{dt^2} = c^2\frac{1}{X}\frac{d^2X}{dx^2} = -\xi^2$$

where $-\xi^2$ is some constant. Recall that we know each term must equal a constant because the left is a function of t only, and the right of x only, and yet they are equal for all values of x and t.

Unlike the previous topic of diffusion, here we find that the both the time and position equations involve second derivatives. We'll find that sine and cosine solutions satisfy them both.

$$\frac{d^2X}{dx^2} = -\xi^2 X \quad \Rightarrow \quad X(t) = A\cos(\xi x) + B\sin(\xi x)$$

while

$$\frac{d^2T}{dt^2} = -c^2\xi^2 T \quad \Rightarrow \quad T(t) = C\cos(\xi ct) + D\sin(\xi ct)$$

so any separable solution is of the form

$$y_{\text{sep}}(x,t) = \left(A\cos(\xi x) + B\sin(\xi x)\right)\left(C\cos(\xi ct) + D\sin(\xi ct)\right).$$

For a general solution we will of course want to combine these separable solutions into a general solution:

$$y(x,t) = \sum_{n=0}^{\infty}\left(A_n\cos(\xi_n x) + B_n\sin(\xi_n x)\right)\left(C_n\cos(\xi_n ct) + D_n\sin(\xi_n ct)\right). \tag{1.11}$$

Remember that we can do this because the wave equation, just like the diffusion equation earlier, is a *linear* equation. Our constants A_n, B_n, C_n, D_n, and especially ξ_n will be determined by the boundary conditions. Whenever we are thinking about a string that is pinned at two points (e.g. at $x = 0$ and $x = L$) then we will find it easy to represent the initial shape of the string (and its initial movement, if any) as a Fourier series, and then we will instantly see the right values to use for the constants. That's because $y(x,t)$ is only *defined* in a finite region, leaving us to define it elsewhere to make it a periodic function.

Following this line of thought, we can simplify the general solution considerably. Let's assume that our string is pinned such that at $x = 0$, the displacement is always zero, i.e. $y(x = 0, t) = 0$. But then, we can always choose to extend $y(x,t)$ into the $x < 0$ region as an *odd* function (and that will be a natural thing to do - why?). In that case we can immediately set all $A_n = 0$.

Furthermore, if the string is also pinned at $x = L$, then this odd function of ours will have period $2L$. Then we can say that $\xi_n = 2n\pi/(2L) = n\pi/L$.

Finally, notice that the velocity of the string is given by

$$\dot{y}(x,t) = c\sum_{n=0}^{\infty} \xi_n \left(A_n \cos(\xi_n x) + B_n \sin(\xi_n x)\right)\left(-C_n \sin(\xi_n ct) + D_n \cos(\xi_n ct)\right).$$

In a problem where the initial velocity $\dot{y}(x, t = 0)$ is zero, we will simply choose all $D_n = 0$.

Using all these simplifications, we can reduce the general solution to just

$$y(x,t) = \sum_{n=0}^{\infty} K_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) \tag{1.12}$$

for some constants K_n to be determined by the initial string shape.

1.8 A plucked string: An example

Suppose that at $t = 0$ our string is pulled into the following **stationary** configuration and released.

$$y(x,t=0) = \frac{2ax}{L} \quad \text{for} \quad 0 \le x \le \frac{L}{2}, \qquad y(x,t=0) = \frac{2a}{L}(L-x) \quad \text{for} \quad \frac{L}{2} \le x \le L.$$

Let's sketch this, and also spot the smart way to extend it to a periodic form:

But this is just the regular triangular function. It's not hard to find the Fourier series. If we consider a simplified function with period 2π which goes between $+1$ and -1, then the Fourier series is

$$\frac{8}{\pi^2}\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin((2n+1)x).$$

Therefore we simply need to scale the amplitude and period (i.e. $2L$):

$$y(x,t=0)=\frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left(\frac{(2n+1)\pi}{L}x\right).$$

Now we need only remember that to make each of these terms match our expression for a separable solution, we must put in the $T(t)$ factor, so

$$y(x,t)=\frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left(\frac{(2n+1)\pi}{L}x\right)\cos\left(\frac{(2n+1)\pi}{L}ct\right).$$

And that's it! What does this equation say about how the plucked string might *sound* as it vibrates?

Matlab exercise 1:
Use matlab to investigate the changing shape of the plucked string.

```
clear;
f=0;
N=10;
L=1;
c=1000;
t=0.0001;
x=0:0.01:1;
for n=0:N
    a=8/(pi^2)*(-1)^n/(2*n+1)^2;
    f = f + a*sin((2*n+1)*pi/L*x)
                        *cos((2*n+1)*pi/L*c*t);
end
myPlot=plot(x,f);
```

Matlab exercise 2:
Use matlab to investigate the sound made by a plucked string. Set x to

some fixed point along the string, and see how the displacement at that point varies in time. Use matlab's **sound( )** function to play it!

```
clear;
f=0;
N=10;
L=1;
c=500;
x=0.5;
t=0:0.0001:0.5;
for n=0:N
    a=8/(pi^2)*(-1)^n/(2*n+1)^2;
    f = f + a*sin((2*n+1)*pi/L*x)
                    *cos((2*n+1)*pi/L*c*t);
end
myPlot=plot(t(1:100),f(1:100));
%% plots just the fist 100 values
sound(f)
```

Matlab exercise 3:
Try the same things as above, but use an initial state for the string where it is plucked away from the centre. The Fourier series for this was found in the Week 2 problem sheet.

1.9 Appendix I: Transforming to the Error Function

Earlier, when we were playing around with finding the most compact solution for the diffusion into a semi-infinite region, in Eqn. (1.8) we used a step that relied on the following equality:

$$\int_0^\infty \frac{\sin(\beta\eta)}{\eta}\exp(-\eta^2)d\eta \;=\; \sqrt{\pi}\int_0^{\beta/2}\exp(-u^2)du.$$

To make these notes complete, its interesting to prove this. The proof we'll use depends on another mathematical fact, a powerful and general principle for performing integrals which involves Fourier transforms! This technique is good to know in its own right.

The Plancherel theorem, due to Michel Plancherel in 1910, states that

$$\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty} F(\xi)G^*(\xi)d\xi$$

if we define the Fourier transform as we did in the first variant in these notes:

$$F(\xi) = \int_{-\infty}^{\infty} f(x)\, \exp(-2\pi i\, \xi x)\, d\xi, \qquad \text{and similarly for } G(\xi).$$

Here the asterisk symbol $*$ has the usual meaning of complex conjugate. This theorem is closely related to Parseval's theorem which we met in Lecture 2.

In the present case, we'll rewrite the theorem using our dummy variables η and u (to avoid confusion with the fact that we are already using x and t!)

$$\int_{-\infty}^{\infty} f(\eta)g^*(\eta)\, d\eta = \int_{-\infty}^{\infty} F(u)G^*(u)du.$$

Now we will choose

$$f(\eta) = \frac{\sin(\beta\eta)}{\eta} \qquad \text{and} \qquad g^*(\eta) = \exp(-\eta^2) \quad \text{and we note } g(\eta) = g^*(\eta).$$

Then we can write $I = \displaystyle\int_0^{\infty} \frac{\sin(\beta\eta)}{\eta} \exp(-\eta^2)d\eta \ = \frac{1}{2}\int_{-\infty}^{\infty} f(\eta)g^*(\eta)d\eta$

since the integrand is even. Now the Fourier transform of $g(\eta) = \exp(-\eta^2)$ is a well known result; with our definition of the Fourier transform, it is

$$G(u) = \sqrt{\pi}\exp(-\pi^2 u^2).$$

We show this at the bottom of the next page, but it's an accepted fact that we can simply use. Meanwhile $F(u)$ is just a 'top hat' function as follows:

$$F(u) = \pi \ \text{ for } \ -\frac{\beta}{2\pi} < u < \frac{\beta}{2\pi}, \qquad F(u) = 0 \ \text{ elsewhere.}$$

The best way to verify this is to remember that the Fourier transform *of a Fourier transform* just gets you back to the original function (up to a scaling factor). So we could take the Fourier transform of this top hat function and it would get us back to $f(\eta)$ correctly. Putting it all together we have,

$$
\begin{aligned}
I = \frac{1}{2}\int_{-\infty}^{\infty} f(\eta)g^*(\eta)d\eta &= \frac{1}{2}\int_{-\infty}^{\infty} F(u)G^*(u)du \\
&= \frac{\pi}{2}\int_{-\beta/2\pi}^{\beta/2\pi} G(u)du \quad \text{using } F(u) \text{ and } G^*(u) = G(u) \\
&= \frac{\pi^{\frac{3}{2}}}{2}\int_{-\beta/2\pi}^{\beta/2\pi} \exp(-\pi^2 u^2)du \quad \text{substituting } G(u) \\
&= \frac{\sqrt{\pi}}{2}\int_{-\beta/2}^{\beta/2} \exp(-v^2)dv \quad \text{with } v \equiv \pi\, u \\
&= \sqrt{\pi}\int_{0}^{\beta/2} \exp(-v^2)dv \quad \text{as its an even function.}
\end{aligned}
$$

That's what we wanted to show!
Let's finish by looking at that 'Fourier transform of a Gaussian' thing:

$$
\begin{aligned}
G(u) &= \int_{-\infty}^{\infty} \exp(-\eta^2)\exp(-2\pi\, i\, \eta u)\, d\eta \\
&= \int_{-\infty}^{\infty} \exp(-\eta^2 - 2\pi i\, u)\, d\eta \\
&= \int_{-\infty}^{\infty} \exp\left(-(\eta + i\pi\, u)^2 - \pi^2 u^2\right)\, d\eta \qquad (1.13) \\
&= \exp(-\pi^2 u^2)\int_{-\infty}^{\infty} \exp\left(-(\eta + i\pi\, u)^2\right)\, d\eta \\
&= \exp(-\pi^2 u^2)\int_{-\infty+i\pi u}^{\infty+i\pi u} \exp(-q^2)\, dq \qquad (1.14) \\
&= \exp(-\pi^2 u^2)\int_{-\infty}^{\infty} \exp(-q^2)dq \qquad (1.15) \\
&= \sqrt{\pi}\exp(-\pi^2 u^2) \qquad (1.16)
\end{aligned}
$$

Here in line (1.13) this is a trick called 'completing the square'. In line (1.14) we change to variable $q \equiv \eta + i\pi u$, and this leads to us adding an imaginary part to the integration limits. In this case it's OK to just drop those imaginary parts from the limits, and we do so in line (1.15) – strictly speaking we must check its OK to do this using something called contour integration. The final line (1.16) is reached when we note that the integral is just some constant, and (although its another little detour to show this...) the constant is $\sqrt{\pi}$.