Fourier Series and PDEs : Problem Sheet 1 of 4

(1) Imagine that you mention to a high school student that you are studying Fourier series, and she asks what they are. Explain in a few lines of text what Fourier series techniques allow us to do. Sketch a graph if that helps. Don't go into how the maths works - just what it achieves.

(2) About even and odd functions.

(a) Suppose two functions $f(x)$ and $g(x)$ are each known to be even. Consider the product function $h(x)$ formed by multiplying f and g together: $h(x) \equiv f(x)g(x)$. Is $h(x)$ even, odd or neither? How about if f and g are both odd, or if one is even and one is odd?

(b) Suppose I have some function $A(x)$ that is defined as a product of N different functions, each of which is known to be either even or odd. (So the previous question above is a special case with $N = 2$.) What do you need to know in order to tell me whether $A(x)$ is odd? Explain your answer.

(c) (Harder, optional) Prove that, in general, any function $f(x)$ which itself is neither even nor odd, can be expressed as the sum of an even and an odd function.

(3) Given that m and n are integers greater than zero, prove the following:

(a) $\int_0^{2\pi} \cos(nx)\cos(mx)dx = 0$ when $m \neq n$.

(b) $\int_0^{2\pi} \cos^2(nx)dx = \pi$.

(c) $\int_0^{2\pi} \sin(nx)\sin(mx)dx = 0$ when $m \neq n$.

(d) $\int_0^{2\pi} \sin^2(nx)dx = \pi$.

(e) $\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0$ for any m, n.
Hint: An elegant way to do this last integral, is to change variable $v \equiv x - \pi$ so that the range of the integral becomes $-\pi$ to π and then try to use your knowledge of even/odd functions.

(4) Using the results of the previous question, write out for yourself the derivation for the Fourier series coefficients: In other words, show that **if it is true that** some periodic function $f(x)$ with period 2π can be written in the Fourier series expansion form,

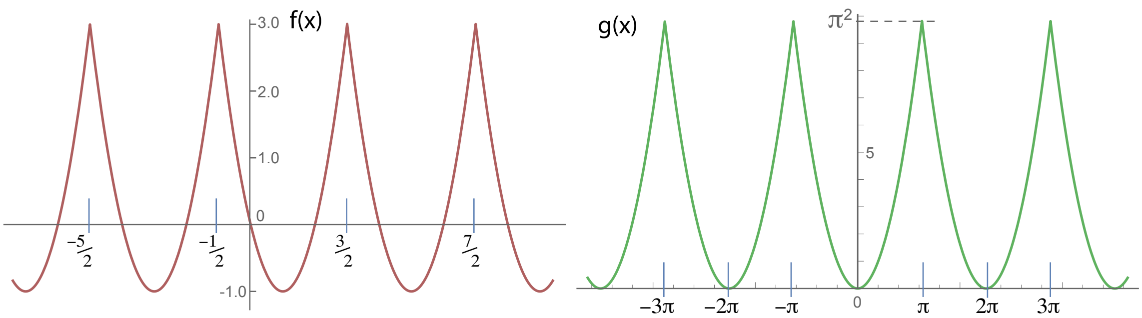
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$\text{then the constants must be } a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx.$$

Note: remember to show that a_0 fits into this rule.

(5) This question is about shifting and scaling. Sketch the simple sine function $f(x) \equiv \sin(x)$, labelling the axes to show the period and the amplitude. Now consider the function $g(x) \equiv A + B\sin(Cx - D)$ where A, B, C and D are constants. Explain in words what effect each of these constants has on shifting and scaling the function; draw four sketches where in each case you start from $A = D = 0$, $B = C = 1$ and show the effect of varying one of the parameters.

(6) Consider the two functions shown in the figure below.



The function on the left is defined by the equations,

$$f(x+2) = f(x) \quad \text{and} \quad f(x) = -1 + 4\left(x - \frac{1}{2}\right)^2 \quad \text{for} \quad \frac{-1}{2} \leq x \leq \frac{3}{2}.$$

(a) Write the equations that define the second function $g(x)$, and derive the Fourier series for this function. You will find it best to do your integrals from $-\pi$ to π, rather than 0 to 2π (remember, any complete cycle will do).

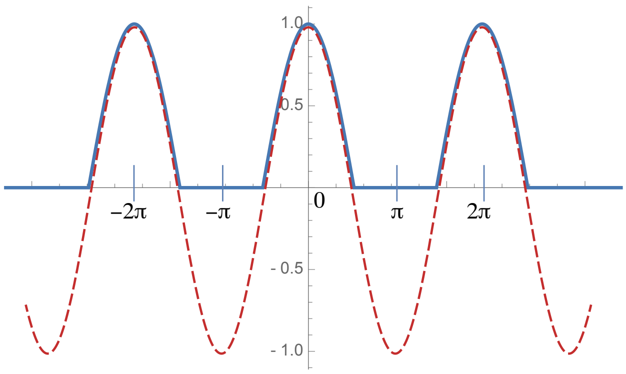
(b) **Matlab task:** Use matlab to confirm that your Fourier series is correct! You can do so by modifying the script that appears in the Week 1 notes.

(c) Carefully figure out how the function $g(x)$ is related to the function $f(x)$. Write your answer by finding the constants A, B, C, D and α, β, γ, δ in the following:

$$f(x) = A + Bg(Cx - D) \qquad g(x) = \alpha + \beta f(\gamma x - \delta)$$

(d) Using your answers to the earlier parts of this question, now write down function $f(x)$ in the form of a sum of cosine terms; those terms can be left written as $\cos(ax - b)$ for suitable a, b.

(7) Challenge question, if you're up for it.
Let's try a more tricky function that is relevant to the real world! Consider the following figure:



Some background information: When a sinusoidally alternating voltage $V = V_0 \cos(t)$ is applied to a simple circuit involving just a resistor, then the current simply varies sinusoidally too. But, if the circuit contains a diode it can block the current from flowing in one direction; then, the current has the form shown in the figure (technically, it is the result of applying a 'half wave rectifier').

(a) Write down the equations that describe the 'rectified' function (the solid blue line).

(b) Is the function even, odd, or neither? What does that mean about the Fourier series?

(c) Derive the Fourier series for this function.
This is more challenging than the derivations we've done before, but give it a go! It will be helpful to work out a_0 and then a_1 separately (quite easy), and then find a_n for $n \geq 2$ (tricky bit!). Here is the answer you should find for that tricky case:

$$\begin{aligned} \text{If } n \geq 2 \text{ then: } a_n &= 0 \quad \text{for odd } n \\ &= (-1)^{\frac{n}{2}} \frac{2}{\pi(1-n^2)} \quad \text{for even } n \end{aligned}$$

(d) **Matlab task:** As before, use matlab to confirm that your Fourier series is correct! You can do so by modifying the script that appears in the Week 1 notes. If you find it hard to generate the complicated coefficients a_n automatically, you can just figure out the first few on paper and type in the function directly – that will give you the approximate shape.