Fourier Series and PDEs : Problem Sheet 2 of 4

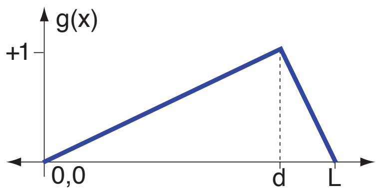
(1) Specify and explain the *Dirichlet conditions* under which a function $g(x)$, which satisfies the periodicity rule $g(x) = g(x+L)$, is guaranteed to have a Fourier series expansion:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2n\pi x/L) + b_n \sin(2n\pi x/L)$$

$$\text{where} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(2n\pi x/L)dx \quad \text{and} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(2n\pi x/L)dx.$$

Matlab task: Invent one or two examples of a function that breaks at least one of the conditions. Use matlab to generate plots of these functions.

(2) Here's a function that could represent the shape of a guitar string when it is plucked at point d, if we assume that the strong is fixed to the guitar at $x = 0$ and $x = L$. Notice that it is ONLY defined in the region $0 < x < L$.



(a) Sketch a function $h(x)$ which is periodic with period L and which coincides with $g(x)$ in the region $0 < x < L$.

(b) Sketch a function $k(x)$ which is periodic with period $2L$, is *even*, and which coincides with $g(x)$ in the region $0 < x < L$.

(c) Sketch a function $q(x)$ which is periodic with period $2L$, is *odd*, and which coincides with $g(x)$ in the region $0 < x < L$.

(d) (This is a bit hard) Have a go at getting a Fourier series expansion for $q(x)$. Put $L = \pi$ to make the total period of $q(x)$ equal to 2π and thus it'll be a bit less horrible! The answer you should get, after a lot of work, is as follows. HINT: Remember (odd)(odd)=(even)

$$q(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \qquad b_n = \frac{2\sin(nd)}{n^2 d(\pi - d)}$$

(3) This question is about the complex form of Fourier series.

(a) Show that, if n and m are integers, then

$$\int_0^{2\pi} \exp(-inx)\exp(imx)dx = 2\pi\delta_{m,n}$$

where $\delta_{m,n}$ is the Kroneker delta which is defined as unity when $m = n$ and zero otherwise.
HINT: One way is to expand the complex exponentials into their $\cos() + i\sin()$ form, and then fully expand out the integral into four parts. Then use the results about cosine and sine integrals you've already proved during Problem Sheet 1. But there are other ways too!

(b) Use the integral from part (a) to show that ***if*** a periodic function $f(x) = f(x+2\pi)$ can be represented as a *complex Fourier series*,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(inx) \qquad \textbf{then it follows that} \qquad c_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(-inx)f(x)dx.$$

(c) Show that if $f(x)$ is real then $c_n^* = c_{-n}$ where the star denotes complex conjugation.

(4) This is about Fourier transforms. Recall the Fourier transform relations for a function $f(x)$

$$f(x) = \int_{-\infty}^{\infty} F(\xi)\exp(2\pi\, i\, \xi x) d\xi \qquad \text{where} \qquad F(\xi) = \int_{-\infty}^{\infty} f(x)\exp(-2\pi\, i\, \xi x) dx$$

(a) Explain how the expression for $f(x)$ is analogous to the expression for the *complex Fourier series* of a function that has period L, as $L \to \infty$.

(b) Now consider the following function $f(x)$

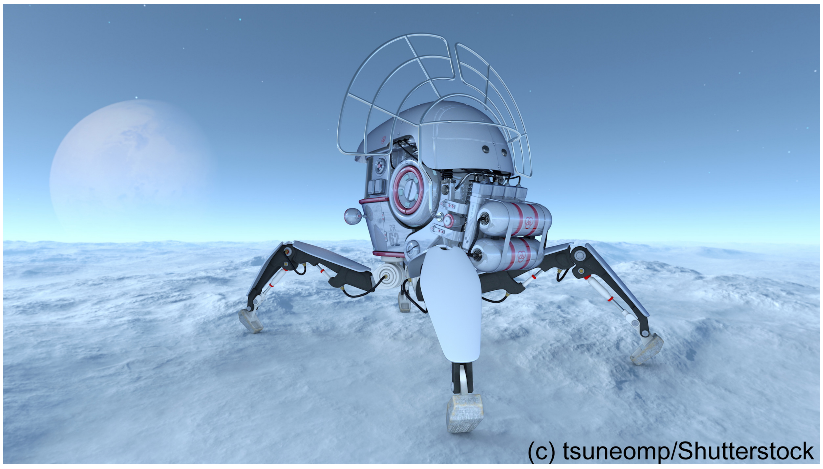
$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad -1 \leq x \leq 1 \\ f(x) &= 0 \quad \text{for} \quad x < -1 \quad \text{and} \quad x > 1. \end{aligned}$$

Draw $f(x)$ over the range $x = -3$ to $x = 3$.

(c) Find $F(\xi)$.

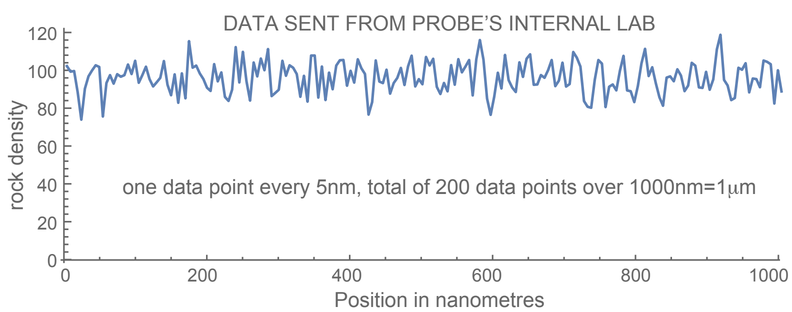
(d) Rewrite the expression we've obtained for $f(x)$ in such a way that there are no complex numbers. HINT Expand the $\exp(ix\alpha)$ term and then use properties of *even* and *odd* functions.

(5) **MATLAB CHALLENGE QUESTION**. Tricky! Feel free to attempt this as a College group.



Background story: A space probe has landed on a comet and obtained a sample of rock. You have a theory that in the rock, there are fossils of tiny organisms. According to your theory there should be a periodic variation in the density of the rock; you are not sure exactly what the period will be, but it should be between 30 and 40 nanometres. You send a signal to the probe, telling it to scan the rock density over a range of one micrometre, recording the density every 5 nanometres. But the data it sends back is extremely noisy. Is there evidence to support your theory? And if so, what is the period?

(a) There is a file called 'spaceData.mlx' on http://ublend.co and on weblearn. Start a matlab session, open and run this script. You should see a plot similar to that below.



The plot shows the varying rock density as the probe's scanner swept over a distance of 1 μm.

(b) Use matlab to perform a Fourier analysis of the data. Try to identify whether there is evidence of a periodic variation, with the period being somewhere between 30 and 40 nanometres.
Be careful: There will be a very strong component with zero frequency (why?) but ignore that and look in right region to find the strongest frequency component(s) of interest.
HINT: Read about the matlab function "fft".