Fourier Series and PDEs : Problem Sheet 3 of 4

(1) A function $G(x,t)$ is governed by the following partial differential equation,

$$\frac{\partial G}{\partial t} = K \frac{\partial^2 G}{\partial x^2}$$

where K is a constant.

(a) Assume that G is *separable* so that $G(x,t) = X(x)\,T(t)$ where $X(x)$ is purely a function of x, and similarly $T(t)$ is purely a function of t. Explaining your steps carefully in your own words, show that the most general solution to the equation can be written

$$G(x,t) = (A\cos(\xi x) + B\sin(\xi x))\exp(-K\xi^2 t)$$

where ξ is any constant.

(b) How can we use this observation to find solutions that are *not* separable?

(2) Suppose that a long thin tube, containing air, suddenly has another gas (call it chlorine, say) introduced. The new gas is not, initially, evenly distributed - so it diffuses though the available space. Suppose that there is no variation in gas concentration across the height and width of the tube, only along its length - then we can write the density of the gas as a function $\phi(x,t)$. By analogy with the heat diffusion example we looked at in the lectures, consider the change in the total mass of gas within a small volume and relate it to the flow in/out. Thus, outline a derivation of the equation governing the diffusion of gas:

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}$$

where D is the diffusivity. You will need to use the equation

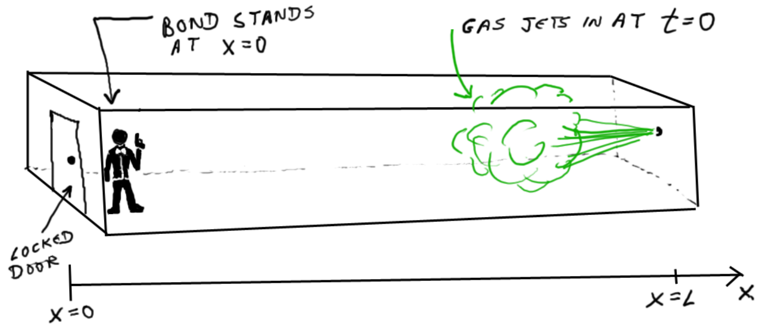
$$f = -k\frac{\partial \phi}{\partial x}$$

where f is now the flow density of gas particles (mass of particles per unit time, per unit perpendicular area) and k is a constant. Collect all constants into D at the end. HINT: But perhaps you'll find that k is the only constant that doesn't cancel out...?

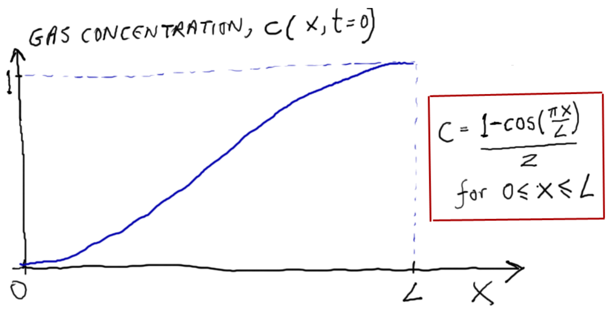
PTO

(3) ***Bond, James Bond!***

Suppose that James Bond is sneaking around the underground headquarters of an evil genius. He enters a long, thin corridor of length L and the door locks shut behind him - sealing him into the corridor. Because he has consumed too many vodka martinis (shaken, not stirred) he doesn't notice a trip wire and, well, trips it. Suddenly a jet of deadly chlorine gas is released into the corridor from the far end. It's only a momentary jet, not a continuous stream, but now it's a race for time as the gas diffuses: 007 must get the door behind him open and escape before the density of gas at his end of the corridor becomes lethal!



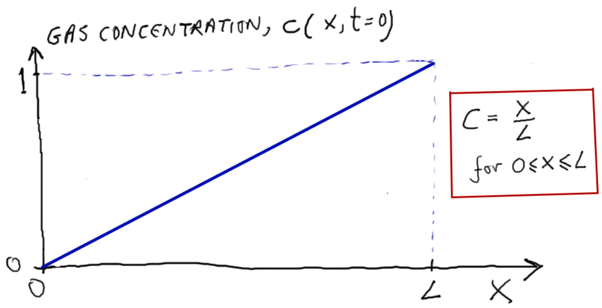
(a) Suppose that, after the initial jet of chlorine at time $t = 0$, it is distributed like this:



Here $c(x, t)$ is the density of gas *relative* to its highest density at $t = 0$, so by definition the maximum of c at $t = 0$ is $c = 1$.
Given that $c(x, t)$ obeys the usual diffusion equation $\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$, work out an expression for the concentration of chlorine $c(x, t)$ along the corridor at any x and t.

(b) Bond can only tolerate chlorine up to 30% of the initial peak density at the other end, i.e. until $c(x = 0, t) = 0.3$. If length $L = 16$ m, and $D = 0.4$ m^2/s, then how long has he got?

(c) Now consider instead the case that the density of gas at $t = 0$ is distributed like this:



Would we expect this to be better or worse for Bond? Work out $c(x, t)$ for this case. You'll need to use a Fourier series; derive the required Fourier coefficients analytically and then use matlab to check that your function is correct.
HINT: You need to extend the function representing the transient part at $t = 0$ to make it periodic, but *don't* use a sawtooth (why not?), instead use a triangular wave.

(d) Using matlab, or by other means, do the following: Again assuming Bond can withstand chlorine density up to $c = 0.3$, and the same L and D as before, estimate how long has he got in this case. How many terms in the Fourier series do you need in order to get 3 decimal places of accuracy?