Fourier Series and PDEs : Problem Sheet 4 of 4

(1) Two blocks of material diffusing into one another at the interface. Suppose that we have two different materials, A and B, in the form of two large slabs. We place the two blocks next to each other so that they have a common planar surface, and then heat them so that the atoms of each material can diffuse into the other block. Assume that the size of the blocks is such that they are effectively infinite in all three dimensions (in comparison to the distance atoms will diffuse in a reasonable time period). Note this situation is not quite the same as in the lecture notes, where we considered *one* block adjacent to a reservoir.

(a) Suppose that, initially, material A occupies the space $x < 0$ and material B occupies $x > 0$ (so that their common surface is the $x = 0$ plane). Let $c(x, t)$ be the concentration of material B, on a scale of zero (pure A) to one (pure B). Sketch $c(x, t = 0)$, the initial distribution, and sketch also the likely distribution at a couple of later times. Assume $c(x = 0, t = 0) = \frac{1}{2}$ since that is the symmetric point in the middle.

(b) What is the value of $c(x, t)$ at any finite x as $t \to \infty$?

(c) You are told that a possible solution is

$$c(x,t) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\beta\right)\right) \qquad \text{where} \qquad \mathrm{erf}(\beta) \equiv \frac{2}{\sqrt{\pi}}\int_0^\beta \exp(-y^2)dy \quad \text{and} \quad \beta \equiv \frac{x}{2\sqrt{Dt}}.$$

Show that this solution is correct by the following two steps:

- Step 1: Check the boundary conditions at $t = 0$ and $t \to \infty$, and at $x = 0$ and $x \to \infty$. You may use the shape of the erf() function, you do not need to derive it.
- Step 2: Substitute the solution $c(x, t)$ into the following diffusion equation, to check that it satisfies:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}.$$

HINT: Variables x and t are in the integral's limit, but you can use a version of Leibniz's integral rule:

$$\frac{d}{d\,a}\int_0^a f(b)\,db = f(a) \quad \text{together with the chain rule} \qquad \frac{\partial g(\beta)}{\partial t} = \frac{d\,g(\beta)}{d\,\beta}\frac{\partial \beta}{\partial t},$$

where here $f()$ and $g()$ are any continuous and differentiable functions.

(2) In this question you will derrive the wave equation for an elastic string, as in the lectures, and then proceed to derive the version for a two dimensional surface like a drum skin.

(a) Suppose that we have an elastic string stretched between two fixed points at $x = 0$ and $x = L$. In your own words, explain how we derive the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}$$

Where $f(x, t)$ is the vertical displacement of the string at horizontal position x and time t, and $c \equiv T/\rho$ for string tension T and density ρ.

(b) Explain why this equation is easily solved if we are told that the solution is *separable*, i.e. of the form $f(x, t) = X(x)T(t)$. Explain how we can find other, non-separable solutions.

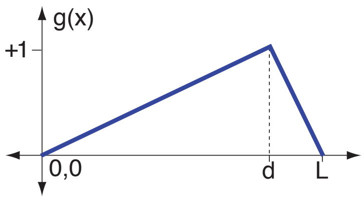
(c) Now consider a stretched rubber sheet, lying in the $z = 0$ plane. Suppose that the sheet is held at $z = 0$ all along some perimeter line, but free to oscillate within this perimeter. Show that the wave equation is now

$$\frac{\partial^2 f}{\partial t^2} = c^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

HINT: As before, work out the net up/down force on a small piece of the elastic sheet.

(d) Show that this 2D case admits separable solutions of the form $f(x, y, t) = X(x)Y(y)T(t)$.

(3) On Problem Sheet 2, Question 2 considered the Fourier series for a function which was like a triangular wave, but with its peak "off centre".



When we solved that problem, we found if $L = \pi$ then $\quad g(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad$ where $\quad b_n = \dfrac{2\sin(nd)}{n^2 d(\pi - d)}$.

(a) Write down the full solution $f(x,t)$ for a string plucked in this way (plucked means that the velocity of the string is zero at $t = 0$, but it has this shape at that moment). Use constant c for the wave velocity.

(b) Suppose that $d = 3\pi/4$. Confirm that the $n = 1$ mode has the greatest amplitude (i.e. b_1 is largest). How does the amplitude and phase of the $n = 2$ mode compare with the first?

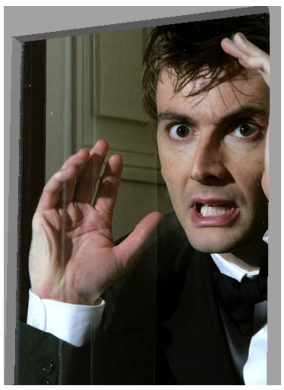
(c) Using matlab, plot the amplitudes of the first 7 different modes (i.e., the constants b_n, for $n = 1...7$) for a range of d from $d = \pi/2$ to $d = 1/100$ (select a few d values from this range). For which value of d will the string sound most like a pure tone, i.e. a tone with a sinusoidal waveform of one frequency?

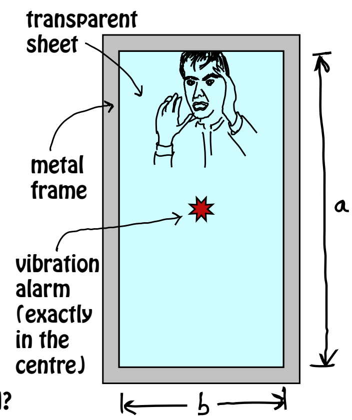
(4) ***Doctor? Doctor Who?***

The cybermen have trapped the tenth Doctor in an alcove behind a sheet of transparent material, while they prepare to "upgrade" him into one of them. Fortunately his companion Martha is still free, and she has his sonic screwdriver! The barrier trapping the Doctor will vibrate according to the 2D wave equation we derived in Q. 2, but unlike rubber it can only bend so much before cracking. So, Martha realizes that she could tune the sonic screwdriver to a resonant frequency and shatter it!


Cyberman - overconfident!


Martha - must find the frequency!


The Doctor - trapped! Doomed?



(a) Suppose that the sheet of material is fixed into a rectangular frame. Show that the frequencies of the vibration modes of the sheet are given by

$$\nu = (c/2)\sqrt{(n/a)^2 + (m/b)^2)}$$

where a and b are the height and width, and c is the velocity of sound in the material.

(b) Suppose that the sheet is twice as tall as it is wide, $a = 2b$. What is the lowest frequency mode of vibration? The cybermen guards in the next room have excellent hearing for high frequencies, but it is safe to use frequencies less than c/b. This means that there are actually three modes that Martha could safely use — show this is the case and draw a sketch showing which part(s) of the sheet vibrate with the greatest amplitude in each case.

(c) Just in time, Martha notices that there is a small device fixed to the exact middle of the sheet – an alarm unit that will trigger if it is vibrated! Is the escape plan foiled... or can the plan still work with the right choice of frequency?